Problem 1. Two metric spaces ρ_1, ρ_2 on X are called *equivalent* if there is a $C > 0$ such that

$$C^{-1}\rho_1(x,y) \le \rho_2(x,y) \le C\rho_1(x,y), \quad \forall x, y \in X. \tag{1}$$

Show that equivalent metric spaces induce the same topology on X. That is, show that $U \subset X$ is open with respect to ρ_1 if and only if it is open with respect to ρ_2.

Proof. ($\implies$) Recall that an open set in a metric space is a subset $U \subset X$ such that for all $x \in U$, there is an $\epsilon > 0$ so that $B_{\rho,\epsilon}(x) = \{y \in X \ : \ \rho(x,y) < \epsilon\} \subset U$. Assume that U is open with respect to ρ_1. Then for all $x \in U$, we have an $\epsilon > 0$ so that $B_{\rho_1,\epsilon}(x) \subset U$. Fix $x \in U$. We need to then find an appropriate $\epsilon' > 0$ so that $B_{\rho_2,\epsilon'}(x) \subset U$.
Taking $\epsilon' = C^{-1}\epsilon$, we see that for all $y \in B_{\epsilon',\rho_2}(x)$ the equivalence (1) gives

$$C^{-1}\rho_1(x,y) \le \rho_2(x,y) < \epsilon' = C^{-1}\epsilon \leftrightarrow \rho_1(x,y) < \epsilon,$$

and so we see that $y \in U$, and so $B_{\rho_2,\epsilon}(x) \subset U$. Since the choice of x was arbitrary, we get that U is open with respect to ρ_2. Hence, we have that if U is open with respect to ρ_1, it is open with respect to ρ_2.
($\impliedby$) Assume that U is open with respect to ρ_2. Then for all $x \in U$, we have that there is an $\epsilon > 0$ so that $B_{\rho_2,\epsilon}(x) \subset U$. Fix $x \in U$. We need to then find an $\epsilon' > 0$ so that $B_{\rho_1,\epsilon'}(x) \subset U$. If we take $\epsilon' = C^{-1}\epsilon$, then we have for all $y \in B_{\rho_1,\epsilon'}(x)$ that

$$\rho_1(x,y) < \epsilon' = C^{-1}\epsilon \leftrightarrow C\rho_1(x,y) < \epsilon,$$

and so the equivalence gives us

$$\rho_2(x,y) \le C\rho_1(x,y) < \epsilon,$$

and so $y \in U$. Hence, $B_{\rho_1,\epsilon'}(x) \subset U$, and since the choice of x was arbitrary we have that U is open with respect to ρ_1. Hence, if U is open with respect to ρ_2, it is open with respect to ρ_1.
Thus, we have that these metrics induce the same topology on X. □

Problem 2. Let (X, ρ) be a metric space.

(1) Let $\alpha : [0,\infty) \to [0,\infty)$ be a continuous, non-decreasing function satisfying:
  - $\alpha(s) = 0$ if and only if $s = 0$, and
  - $\alpha(s+t) \le \alpha(s) + \alpha(t)$ for all $s, t \ge 0$.

  Define $\sigma(x,y) := \alpha(\rho(x,y))$. Show that σ is a metric, and σ induces the same topology on X as ρ.

(2) Define $\rho_1, \rho_2 : X \times X \to [0,\infty)$ by

$$\rho_1(x,y) := \begin{cases} \rho(x,y) \text{ if } \rho(x,y) \le 1 \\ 1 \text{ otherwise.} \end{cases}$$

$$\rho_2(x,y) := \frac{\rho(x,y)}{1+\rho(x,y)}.$$

  Use (1) to show that ρ_1 and ρ_2 are metric spaces on X which induce the same topology on X as ρ.

Proof. (1) We first establish that σ is a metric. Recall that σ is a metric if it satisfies four properties:

  (a) $\sigma(x,y) \ge 0$ for all $x, y \in X$.
  (b) $\sigma(x,y) = 0$ if and only if $x = y$ for all $x, y \in X$.
  (c) $\sigma(x,y) = \sigma(y,x)$ for all $x, y \in X$.
  (d) $\sigma(x,z) \le \sigma(x,y) + \sigma(y,z)$ for all $x, y, z \in X$.

To see (a), we see that ρ is a metric and so satisfies (a), and furthermore α is non-negative, so

$$\sigma(x,y) = \alpha(\rho(x,y)) \geq 0$$

for all $x, y \in X$.
To see (b), we start with the implication. If $\sigma(x,y) = 0$, then we have $\alpha(\rho(x,y)) = 0$. Since $\alpha(x) = 0$ if and only if $x = 0$, this implies that $\rho(x,y) = 0$, and since ρ is a metric we must have that $x = y$. For the converse, if $x = y$, then $\rho(x,y) = 0$, $\alpha(0) = 0$, and so $\sigma(x,y) = 0$.
To see (c), we have α is well-defined, so if $x = y$ we get $\alpha(x) = \alpha(y)$. Hence, using the fact that $\rho(x,y) = \rho(y,x)$ since ρ is a metric, we have

$$\sigma(x,y) = \alpha(\rho(x,y)) = \alpha(\rho(y,x)) = \sigma(y,x)$$

for all $x, y \in X$.
Finally, to see (d), take $x, y, z \in X$. We see that

$$\sigma(x,z) = \alpha(\rho(x,z)).$$

Since ρ is a metric, we have

$$\rho(x,z) \leq \rho(x,y) + \rho(y,z).$$

Notice that α is non-decreasing, so

$$\alpha(\rho(x,z)) \leq \alpha(\rho(x,y) + \rho(y,z)).$$

Finally, α is subadditive, so

$$\alpha(\rho(x,y) + \rho(y,z)) \leq \alpha(\rho(x,y)) + \alpha(\rho(y,z)) = \sigma(x,y) + \sigma(y,z).$$

So we have for all $x, y, z \in X$,

$$\sigma(x,z) \leq \sigma(x,y) + \sigma(y,z).$$

Thus, we see σ satisfies properties (a)-(d), and so it is a metric.

We now wish to establish that σ induces the same topology as ρ on X. That is, U is open with respect to σ if and only if it is open with respect to ρ.
($\implies$) Assume that U is open with respect to σ. Then we have for all $x \in U$, there is an $\epsilon > 0$ so that $B_{\sigma,\epsilon}(x) \subset U$. That is, if $y \in B_{\sigma,\epsilon}(x)$, we have

$$\sigma(x,y) = \alpha(\rho(x,y)) < \epsilon.$$

Take $\lambda \in (0, \epsilon)$ such that $\alpha^{-1}(\lambda) = \{z \in [0, \infty) \ : \ \alpha(z) = \lambda\} \neq \varnothing$; such a λ exists, since α is continuous and $\alpha(s) = 0$ if and only if $s = 0$. Take $\epsilon' \in \alpha^{-1}(\lambda)$. Then we have that if y is such that

$$\rho(x,y) < \epsilon',$$

that is, $y \in B_{\rho,\epsilon'}(x)$, then

$$\sigma(x,y) = \alpha(\rho(x,y)) \leq \alpha(\epsilon') = \lambda < \epsilon$$

by the monotonicity of α, and so $y \in B_{\sigma,\epsilon}(x) \subset U$. Hence, $B_{\rho,\epsilon'}(x) \subset U$.
($\impliedby$) Assume U is open with respect to ρ. Taking $x \in U$, we have that there is an $\epsilon > 0$ so that $B_{\rho,\epsilon}(x) \subset U$. Take $\epsilon' = \alpha(\epsilon) > 0$. Then we have that if y is such that

$$\sigma(x,y) = \alpha(\rho(x,y)) < \epsilon' = \alpha(\epsilon),$$

then we must have

$$\rho(x,y) < \epsilon,$$

since α is non-decreasing. Hence, we get $B_{\sigma,\epsilon'}(x) \subset U$. So these two metrics induce the same topology on X.

(2) Let

$$\alpha_1(x) := \begin{cases} x \text{ if } x \leq 1, \\ 1 \text{ otherwise.} \end{cases}$$

$$\alpha_2(x) := \frac{x}{1+x}$$

We need to establish that $\alpha_1, \alpha_2 : [0, \infty) \to [0, \infty)$ are **well-defined** functions which are **continuous**, **non-decreasing**, **subadditive**, and which satisfy $\alpha_i(x) = 0$ **if and only** $x = 0$. We note here that it's clear that the image of α_1 will be in $[0, \infty)$, since $x \in [0, 1]$ maps to x and $x > 1$ maps to 1. Likewise, we see that the image of α_2 will be in $[0, \infty)$, since if $x \in [0, \infty)$, then $1 + x \in [1, \infty)$, and so $x/(1 + x) \in [0, \infty)$ as well. So this condition is satisfied.

Well-defined: (α_1): If $x = y \in [0, 1]$, then we have that $\alpha_1(x) = x = y = \alpha_1(y)$. If $x = y \in (1, \infty)$, we see that $\alpha_1(x) = 1 = \alpha_1(y)$. So the function is well-defined.

(α_2): If $x = y \in [0, \infty)$, then $1 + x = 1 + y$ as well. So $\alpha_2(x) = x/(1 + x) = y/(1 + y) = \alpha_2(y)$, and the function is well-defined.

Continuity: (α_1): For all $\epsilon > 0$, if we take $\delta = \epsilon$, we have that $|x - y| < \delta$ implies $|\alpha_1(x) - \alpha_1(y)| < \epsilon$. To see this, let $x < y$, $x, y \in [0, \infty)$ throughout. We break it up into cases based on where x and y are. If $x, y \in [0, 1]$, then we have

$$|\alpha_1(x) - \alpha_1(y)| = |x - y| < \epsilon,$$

and so we are done. If $x \in [0, 1]$, $y \in (1, \infty)$, we have $\alpha_1(y) < y$, and so

$$|\alpha_1(y) - \alpha_1(x)| = \alpha_1(y) - \alpha_1(x) < y - x = |x - y| < \epsilon,$$

and so we are done. If $x, y \in (1, \infty)$, then we have

$$|\alpha_1(y) - \alpha_1(x)| = 0 < \epsilon.$$

Since these are all the possibilities, we see that $\alpha_1(x)$ is continuous.

(α_2): Recall from undergraduate analysis that if f, g are continuous, $g \neq 0$ on the domain, then f/g is also continuous. Notice that the function x is clearly continuous, adding 1 to it is still continuous, and since $1 + x \neq 0$ for all $x \in [0, \infty)$, we have that α_2 is continuous.

Non-decreasing: (α_1): Again, this is clear. If $x, y \in [0, \infty)$, $x < y$, then we have $\alpha_1(x) \leq \alpha_1(y)$. To see this, break it up into cases again; if $x, y \in [0, 1]$, then $\alpha_1(x) = x \leq \alpha_1(y) = y$; if $x \in [0, 1]$, $y \in (1, \infty)$, we have $\alpha_1(x) = x \leq \alpha_1(y) = 1$ by definition; if $x, y \in (1, \infty)$, then $\alpha_1(x) = \alpha_1(y)$.

(α_2): If $x < y$, $x, y \in [0, \infty)$, we have

$$\frac{y}{1+y} - \frac{x}{1+x} = \frac{y - x}{(x+1)(y+1)}.$$

Since $y > x$, we have $y - x > 0$, and since $x, y > 0$, $1 + x, 1 + y > 0$. So, we get that

$$\frac{y}{1+y} - \frac{x}{1+x} > 0 \leftrightarrow \alpha_2(y) > \alpha_2(x),$$

so this is non-decreasing.

Subadditive: (α_1): Take $x, y \in [0, \infty)$, $x \leq y$. We can break this up into cases by x, y, and $x + y \in [0, \infty)$.

If $x, y \in [0, 1]$, $x + y \in [0, 1]$, then we have

$$\alpha_1(x + y) = x + y = \alpha_1(x) + \alpha_2(y).$$

If $x, y \in [0, 1]$, $x + y \in (1, \infty)$, then we have

$$\alpha_1(x + y) = 1 \leq x + y = \alpha_1(x) + \alpha_1(y).$$

If $x \in [0,1]$, $y, x+y \in (1,\infty)$, we have

$$\alpha_1(x+y) = 1 \leq x+1 = \alpha_1(x) + \alpha_1(y).$$

If $x, y, x+y \in (1,\infty)$, we have

$$\alpha_1(x+y) = 1 \leq 2 = \alpha_1(x) + \alpha_1(y).$$

Since these are all the possibilities, we have that α_1 is subadditive.
(α_2): Notice that, for $x, y \in [0,\infty)$, we have

$$\alpha_2(x+y) = \frac{x+y}{1+x+y},$$

$$\alpha_2(x) + \alpha_2(y) = \frac{x}{1+x} + \frac{y}{1+y} = \frac{2xy+x+y}{xy+x+y+1}.$$

Subtracting, we have

$$\alpha_2(x) + \alpha_2(y) - \alpha_2(x+y) = \frac{xy(x+y+2)}{(x+1)(y+1)(x+y+1)}.$$

Since $x, y \geq 0$, we get that this is greater than or equal to 0, and so

$$\alpha_2(x) + \alpha_2(y) \geq \alpha_2(x+y).$$

$\alpha_i(x) = 0$ if and only if $x = 0$: (α_1): This follows by definition.
(α_2): Algebra gives us that

$$\frac{x}{1+x} = 0 \leftrightarrow x = 0.$$

Since all the conditions from (1) are satisfied, we have that $\rho_1 := \alpha_1 \circ \rho$, $\rho_2 := \alpha_2 \circ \rho$ are metrics which induce the same topology on X as ρ.

□

Problem 3. A collection of subsets $\{F_i\}_{i\in I}$ of X has the *finite intersection property* (abbreviated FIP) if, for any finite $J \subset I$, we have

$$\bigcap_{j\in J} F_j \neq \varnothing.$$

Prove that for a metric (or topological) space, the following are equivalent:

(1) Every open cover of X has a finite subcover.
(2) For every collection of closed subsets $\{F_i\}_{i\in I}$ with the finite intersection property,

$$\bigcap_{i\in I} F_i \neq \varnothing.$$

Proof. (1) $\implies$ (2): We proceed by contradiction. Assume that every open cover of X has a finite subcover, $\{F_i\}_{i\in I}$ is a collection of closed subsets satisfying the finite intersection property. Assume that

$$\bigcap_{i\in I} F_i = \varnothing.$$

DeMorgan's then gives

$$\left(\bigcap_{i\in I} F_i\right)^C = \bigcup_{i\in I} F_i^C = X.$$

Since the F_i are closed, we have that the $U_i = F_i^C$ are open. Since X is compact, we can take a finite subcover to get

$$\bigcup_{i \in J} U_i = X,$$

but applying DeMorgan's again gives

$$\bigcap_{i \in J} F_i = \varnothing,$$

contradicting the FIP assumption. Hence, we must have that

$$\bigcap_{i \in I} F_i \neq \varnothing.$$

(2) $\implies$ (1): We proceed by contradiction again. Take an open cover of X;

$$\bigcup_{i \in I} U_i = X.$$

DeMorgan's gives us

$$\bigcap_{i \in I} F_i = \varnothing,$$

where $F_i = U_i^C$ are closed subsets. If X does not admit a a finite refinement of the cover $\{U_i\}_{i \in I}$, then we have that there are no $J \subset I$ finite so that

$$\bigcap_{i \in J} F_i = \varnothing;$$

but this tells us that $\{F_i\}_{i \in I}$ has the FIP, and so by assumption we have

$$\bigcap_{i \in I} F_i \neq \varnothing,$$

a contradiction again. Hence, we must have that there is some $J \subset I$ finite so that

$$\bigcup_{i \in J} U_i = X.$$

That is, X is compact. $\square$

Problem 4. Let X be a set. A *π-system* on X is a collection of subsets $\Pi \subset P(X)$ which is closed under finite intersections. A *λ-system* on X is a collection of subsets $\Lambda \subset P(X)$ such that

- $X \in \Lambda$,
- Λ is closed under taking complements, and
- for every sequence of disjoint subsets $\{E_i\}$ in Λ, $\bigcup E_i \in \Lambda$.

(1) Show that $\mathcal{M}$ is a σ-algebra if and only if $\mathcal{M}$ is both a π-system and a λ-system.
(2) Suppose Λ is a λ-system. Show that for every set $E \in \Lambda$, the set

$$\Lambda(E) = \{F \subset X \ : \ F \cap E \in \Lambda\}$$

is also a λ-system.

Proof. (1) ($\implies$) Assume that $\mathcal{M}$ is a σ-algebra. Then recall that this means that

- $X \in \mathcal{M}$,
- $\mathcal{M}$ is closed under complements,
- $\mathcal{M}$ is closed under countable unions.

We first want to establish that $\mathcal{M}$ is a π-system. If $\mathcal{M}$ is closed under complements and countable unions, then DeMorgans gives us that it is closed under countable intersections, and more specifically under finite intersections. Thus, $\mathcal{M}$ is a π-system.
We then want to establish that $\mathcal{M}$ is a λ-system. This just follows from the following:

- $X \in \mathcal{M}$, since $\mathcal{M}$ is a σ-algebra,
- $\mathcal{M}$ is closed under complements, since $\mathcal{M}$ is a σ-algebra,
- $\mathcal{M}$ is closed under countable unions, since $\mathcal{M}$ is a σ-algebra, and so more specifically closed under countable unions of disjoint subsets.

Thus, $\mathcal{M}$ is a λ-system.
($\Longleftarrow$) If $\mathcal{M}$ is a λ-system and a π-system, then we get for free that $X \in \mathcal{M}$ and $\mathcal{M}$ is closed under complements. It remains to check that $\mathcal{M}$ is closed under countable unions. That is, if we let $\{E_i\}_{i=1}^{\infty}$ be a collection of sets in $\mathcal{M}$, then we want to show that

$$\bigcup_{i=1}^{\infty} E_i \in \mathcal{M}.$$

Construct a sequence of disjoint sets as follows: let $H_1 = E_1$, and

$$H_i = E_i \cap E_{i-1}^C \cap \cdots \cap E_1^C.$$

Then it's clear that, for all $i \neq j$, we have

$$H_i \cap H_j = \varnothing.$$

Furthermore, we have that $H_i \in \mathcal{M}$ for all i; this is because it is closed under complements (from being a λ-system) and finite intersections (from being a π-system). So we have

$$\bigcup_{i=1}^{\infty} H_i \in \mathcal{M},$$

and by construction we have

$$\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} H_i,$$

so

$$\bigcup_{i=1}^{\infty} E_i \in \mathcal{M}.$$

Hence, $\mathcal{M}$ is closed under countable unions, and so a σ-algebra.

(2) Let $E \in \Lambda$ be arbitrary. First, notice that $F \cap X = F \in \Lambda$, so $X \in \Lambda(E)$.
Next, if $F \in \Lambda(E)$, we want to show that $F^C \in \Lambda(E)$; that is, $F^C \cap E \in \Lambda$. Using DeMorgan's laws, and noticing that $E \cap E^C = \varnothing$, we can write this as

$$E \cap F^C = (E \cap F^C) \cup (E \cap E^C) = E \cap (F \cap E)^C = (E^C \cup (F \cap E))^C$$

Now, $F \cap E, E \in \Lambda$ by assumption, $E^C \in \Lambda$ since it is closed under complements, and notice that $E^C \cap (F \cap E) = \varnothing$; that is, they are disjoint. Thus, we have that $E^C \cup (F \cap E) \in \Lambda$. Again, using the fact that Λ is closed under complements, we have $(E^C \cup (F \cap E))^C \in \Lambda$, but this translates to $F^C \cap E \in \Lambda$. Hence, $\Lambda(E)$ is closed under complements.
Finally, we need to show that it is closed under disjoint unions. Let $\{F_i\}$ be a collection of disjoint sets in $\Lambda(E)$. Then we have, for all i, $F_i \cap E \in \Lambda$. Furthermore, since $F_i \cap F_j = \varnothing$ for $i \neq j$, we have $(F_i \cap E) \cap (F_j \cap E) = \varnothing$. So,

$$\bigcup (F_i \cap E) = \Big(\bigcup F_i\Big) \cap E \in \Lambda,$$

since Λ is closed under disjoint unions, and so

$$\bigcup F_i \in \Lambda(E).$$

Hence, $\Lambda(E)$ is closed under disjoint unions. Since it satisfies the three properties, we get that $\Lambda(E)$ is a λ-system.

□

Problem 5. Let Π be a π-system, let Λ be the smallest λ-system containing Π, and let $\mathcal{M}$ be the smallest σ-algebra containing Π.

(1) Show that $\Lambda \subset \mathcal{M}$.
(2) Show that for every $E \in \Pi$, $\Pi \subset \Lambda(E)$, where $\Lambda(E)$ was defined in the problem above. Deduce that $\Lambda \subset \Lambda(E)$ for every $E \in \Pi$.
(3) Show that $\Pi \subset \Lambda(F)$ for every $F \in \Lambda$. Deduce that $\Lambda \subset \Lambda(F)$ for every $F \in \Lambda$.
(4) Deduce that Λ is a σ-algebra, and thus $\mathcal{M} = \Lambda$.

Proof. (1) From **Problem 4 (1)**, we have that $\mathcal{M}$ is a π-system and a λ-system. Since Λ is the smallest λ-system containing Π by assumption, we get that $\Lambda \subset \mathcal{M}$.

(2) Fix $E \in \Pi$ and take $F \in \Pi$. Since Π is closed under finite intersection, we get that $F \cap E \in \Pi \subset \Lambda$. But since $F \cap E \in \Lambda$, we must have $F \in \Lambda(E)$. Since F was arbitrarily chosen in Π, we get that $\Pi \subset \Lambda(E)$ for all $E \in \Pi$. From **Problem 4 (2)**, we know that $\Lambda(E)$ is a λ-system, and since Λ is the smallest λ-system containing Π, we must have $\Lambda \subset \Lambda(E)$ for all $E \in \Pi$.

(3) Fix $F \in \Lambda$ and take $E \in \Pi$. Since $\Lambda \subset \Lambda(E)$ from (2), we have that for all $G \in \Lambda$, $E \cap G \in \Lambda$. Hence, in particular, we have $E \cap F \in \Lambda$, but this implies that $E \in \Lambda(F)$. Thus, $\Pi \subset \Lambda(F)$. Since $\Lambda(F)$ is a λ-system by **Problem 4 (2)**, this implies that $\Lambda \subset \Lambda(F)$ by minimality.

(4) Notice that for all $F \in \Lambda$, we have $\Lambda \subset \Lambda(F)$. In other words, for all $E, F \in \Lambda$, we get $E \cap F \in \Lambda$. We can then extend this to finite intersections, and so we have that Λ is closed under finite intersections; in other words, Λ is a π-system. Since Λ is both a π-system and a λ-system, **Problem 4 (1)** tells us that Λ is a σ-algebra. Since $\mathcal{M}$ is the smallest σ-algebra containing Π, we get $\mathcal{M} \subset \Lambda$. Coupling this with (1) from this problem, we have that $M = \Lambda$.

□

Remark. Thomas O'Hare was a collaborator for this.

Problem 6. Let Π be a π-system, and let $\mathcal{M}$ be the smallest σ-algebra containing Π. Suppose μ and ν are two measures on $\mathcal{M}$ whose restriction on Π agree.

(1) Show that if μ and ν are finite and $\mu(X) = \nu(X)$, then $\mu = \nu$.
(2) Suppose that $X = \bigsqcup_{j=1}^{\infty} X_j$ with $(X_j) \subset \Pi$ and $\mu(X_j) = \nu(X_j) < \infty$ for all $J \in \mathbb{N}$. Show that $\mu = \nu$.

Proof. (1) We proceed via the hint. Consider $\Lambda := \{E \in \mathcal{M} \ : \ \nu(E) = \mu(E)\}$. By assumption, $\Pi \subset \Lambda$, and we'd like to show that Λ is a λ-system. If we do so, we get that $\mathcal{M} \subset \Lambda$ by the prior homework, which implies that μ and ν are equal on all of $\mathcal{M}$.

Recall that a λ-system needs to satisfy three things:

- $X \in \Lambda$;
- Λ is closed under taking complements;
- For every sequence of disjoint sets $\{E_i\}$ in Λ, $\bigcup E_i \in \Lambda$.

First, we want to show that it's closed under disjoint unions. Let $\{E_i\}$ be a sequence of disjoint sets in Λ. Then we have

$$\mu\left(\bigsqcup E_i\right) = \sum \mu(E_i) = \sum \nu\left(E_i\right) = \nu\left(\bigsqcup E_i\right)$$

since μ and ν are measures, and we have countable additivity. So $\bigsqcup E_i \in \Lambda$.

Next, it follows that $X \in \Lambda$ since $\mu(X) = \nu(X) < \infty$.

Finally, we want to show that it's closed under complements. Let $A \in \Lambda$. Since $A \sqcup A^C = X$ and $\mu(X) = \nu(X) < \infty$, we have that

$$\mu(A) + \mu(A^C) = \mu(A \sqcup A^C) = \mu(X) = \nu(X) = \nu(A \sqcup A^C) = \nu(A) + \nu(A^C),$$

and since $\mu(A) = \nu(A) < \infty$ we can subtract them from both sides to get $\mu(A^C) = \nu(A^C)$. Hence, $A^C \in \Lambda$.

Remark. Prior to Dr. Penneys fixing the problem, I had done this with an increasing cover. To remediate this, let $E_n = \sqcup_{i=1}^{n} X_i$. Then we have an increasing cover $E_n \nearrow X$ and the argument below applies.

(2) We do the same trick, letting $\Lambda = \{E \in \mathcal{M} \ : \ \mu(E) = \nu(E)\}$. Again, we get that it's closed under disjoint unions by the same argument from prior.

Next, to see that $X \in \Lambda$, we get that

$$\mu(X) = \lim_{n\to\infty} \mu(E_i) = \lim_{n\to\infty} \nu(E_i) = \nu(X),$$

using the continuity from below property.

Finally, we want to show that if $A \in \Lambda$, then $A^C \in \Lambda$. Notice that we have

$$\mu\left((A \sqcup A^C) \cap \bigcup_{i=1}^{N} E_i\right) = \mu\left(\bigcup_{i=1}^{N}(A \cap E_i) \sqcup \bigcup_{i=1}^{N}(A^C \cap E_i)\right) = \mu\left(\bigcup_{i=1}^{N}(A \cap E_i)\right) + \mu\left(\bigcup_{i=1}^{N}(A^C \cap E_i)\right)$$

$$= \nu\left((A \sqcup A^C) \cap \bigcup_{i=1}^{N} E_i\right) = \nu\left(\bigcup_{i=1}^{N}(A \cap E_i) \sqcup \bigcup_{i=1}^{N}(A^C \cap E_i)\right) = \nu\left(\bigcup_{i=1}^{N}(A \cap E_i)\right) + \nu\left(\bigcup_{i=1}^{N}(A^C \cap E_i)\right).$$

By assumption,

$$\mu\left(\bigcup_{i=1}^{N}(A \cap E_i)\right) = \nu\left(\bigcup_{i=1}^{N}(A \cap E_i)\right) < \infty,$$

so we can subtract it from both sides to get

$$\mu\left(\bigcup_{i=1}^{N}(A^C\cap E_i)\right)=\nu\left(\bigcup_{i=1}^{N}(A^C\cap E_i)\right).$$

Since this applies for all N, we can use continuity from below to get that

$$\mu\left(\bigcup_{i=1}^{\infty}(A^C\cap E_i)\right)=\lim_{n\to\infty}\mu(A^C\cap E_n)=\mu(A^C)=\nu(A^C)=\lim_{n\to\infty}\nu(A^C\cap E_n)=\nu\left(\bigcup_{i=1}^{\infty}(A^C\cap E_i)\right).$$

Hence, we get $A^C\in\Lambda$. So Λ is indeed a λ-system.

□

Problem 7 (Folland 1.14, Folland 1.15)**.** Given a measure μ on $(X,\mathcal{M})$, define ν on $\mathcal{M}$ by

$$\nu(E):=\sup\{\mu(F)\ :\ F\subset E \text{ and } \mu(F)<\infty\}.$$

(1) Show that ν is a semifinite measure. We call it the *semifinite part* of μ.
(2) Suppose $E\in\mathcal{M}$ with $\nu(E)=\infty$. Show that for any $n>0$, there is an $F\subset E$ such that $n<\nu(F)<\infty$.
(3) Show that if μ is semifinite, then $\mu=\nu$.
(4) Show there is a measure ρ on $\mathcal{M}$ which assumes only the values 0 and ∞ such that $\mu=\nu+\rho$.

Proof. (1) We need to show the ν is a measure. That is, it satisfies two properties:
(i) $\nu(\varnothing)=0$;
(ii) if $\{E_i\}\subseteq\mathcal{M}$ disjoint, then

$$\nu\left(\bigcup E_i\right)=\sum\nu(E_i).$$

For (i), we see clearly $\nu(\varnothing)=0$.
For (ii), let $\{E_i\}$ be a sequence of disjoint sets in $\mathcal{M}$. We see

$$\nu\left(\bigsqcup E_i\right)=\sup\left\{\mu(F)\ :\ F\subset\bigsqcup E_i \text{ and } \mu(F)<\infty\right\}.$$

Since these are disjoint, we see that the $F\subset\bigsqcup E_i$ can be seen as a disjoint union of $F_i\subset E_i$ by setting $F_i=(F\cap E_i)$; that is, $F=\bigsqcup F_i\subset\bigsqcup E_i$. Using the fact that μ is a measure and the F_i are disjoint, we can write this as

$$\nu\left(\bigsqcup E_i\right)=\sup\left\{\sum\mu(F_i)\ :\ F_i\subset E_i \text{ and } \mu(F_i)<\infty \text{ for all } i\right\}.$$

Notice that this is equal to

$$\sum\sup\{\mu(F_i)\ :\ F_i\subset E_i \text{ and } \mu(F_i)<\infty\}=\sum\nu(E_i).$$

So this is indeed a measure.

We need to now show that ν is semifinite. Recall that ν is a semifinite measure if, for all $E\in\mathcal{M}$ with $\nu(E)=\infty$, there exists an $F\in\mathcal{M}$ such that $F\subset E$ and $0<\nu(F)<\infty$.

Take $E\in\mathcal{M}$ where $\nu(E)=\infty$. Then this is equivalent to saying that

$$\sup\{\mu(F)\ :\ F\subset E \text{ and } \mu(F)<\infty\}=\infty.$$

But this then implies that there is an $F\subset E$ so that $0<\mu(F)<\infty$ (the empty set will always be in this set, and if the supremum is infinity we must have something which is not 0 since the supremum is the least upper bound). We see that

$$\nu(F)=\sup\{\mu(G)\ :\ G\subset F \text{ and } \mu(G)<\infty\},$$

and the monotonicity of μ tells us that any such $\mu(G)\le\mu(F)<\infty$. It follows that the supremum will be $\mu(F)$ and therefore $0<\nu(F)<\infty$. Thus, ν is a semifinite measure.

(2) Assume otherwise. That is, let $n = \sup\{F \ : \ F \subset E,\ F \in \mathcal{M},\ 0 < \nu(F) < \infty\}$. Then we have that $n < \infty$ by assumption. Since this is a supremum property, for each $k \geq 1$ an integer we have that there are corresponding sets F_k so that

$$n - 1/k < \nu(F_k) \leq n.$$

Let $H_m = \bigcup_{k=1}^{m} F_k$. This is a measurable set, since $\mathcal{M}$ is a σ-algebra. Notice that $F_k \subset H_m$ for all $1 \leq k \leq m$, and so we get that for all such k,

$$n - 1/k < \nu(H_m) \leq n.$$

Taking a union over all m, calling this set H, we get an increasing sequence, and so we can use continuity from below to get that

$$n - 1/k < \nu(H) \leq n$$

for all k, and so therefore

$$\nu(H) = n$$

and $H \subset E$. Now, since these are measurable sets, we can use the measure property of ν to write

$$\nu(E) = \nu(E \cap H) + \nu(E \cap H^C).$$

Since $\nu(E) = \infty$, $H \subset E$, we get that

$$\infty = n + \nu(E \cap H^C),$$

or in other words,

$$\nu(E - H) = \infty.$$

Now, ν is a semifinite measure, so we have that there is a measurable set, say G, so that $G \subset E - H$ and $0 < \nu(G) < \infty$. Since $G \subset E - H$, we have that it is disjoint from H, and so we have

$$n < \nu(H \cup G) = \nu(H) + \nu(G) < \infty.$$

Notice as well that $H \subset E$, $G \subset E$, so we have that $H \cup G \subset E$. But this implies that $H \cup G \subset \{F \ : \ F \subset E,\ F \in \mathcal{M},\ 0 < \nu(F) < \infty\}$, and so therefore we have

$$\sup\{F \ : \ F \subset E,\ F \in \mathcal{M},\ 0 < \nu(F) < \infty\} \geq \nu(H \cup G) > n,$$

which is a contradiction. Thus, we cannot have such an n.

(3) We want to show that, for all $E \in \mathcal{M}$, we have $\mu(E) = \nu(E)$. If $\mu(E) < \infty$, then the proof in (1) establishes that $\mu(E) = \nu(E)$. If $\mu(E) = \infty$, then the proof in (2) applies (since we didn't use any property of μ in this proof), and so we have for all $n > 0$, there is an $F \subset E$ so that $n < \mu(F) < \infty$. Hence, we get that

$$\nu(E) = \sup\{\mu(F) \ : \ F \subset E \text{ and } \mu(F) < \infty\} = \infty,$$

and so $\mu(E) = \nu(E)$. We conclude that $\mu = \nu$.

(4) Throughout, let σ-finite denote μ-σ-finite. Let $\rho : \mathcal{M} \to [0, \infty]$ be defined by

$$\rho(E) = \begin{cases} 0 \text{ if } E \text{ is } \sigma\text{-finite} \\ \infty \text{ if } E \text{ is not } \sigma\text{-finite.} \end{cases}$$

It's clear to see that $\rho(\varnothing) = 0$. Next, we check countable additivity. We show this by breaking it up into cases

Case 1: Assume $\bigsqcup E_n$ is σ-finite. This tells us that there is a disjoint collection $\{K_i\}$ such that $\mu(K_i) < \infty$ for all i and $\bigsqcup K_i = \bigsqcup E_n$. Taking E_i arbitrary, we have that $E_i \subset \bigsqcup K_i$, and so $E_i = \bigsqcup (K_j \cap E_i)$, $\mu(K_j \cap E_i) \le \mu(K_j) < \infty$, and thus we have that E_i is σ-finite. So, the following formula applies:

$$0 = \rho\left(\bigsqcup E_n\right) = \sum \rho(E_n) = 0.$$

Case 2: Assume $\bigsqcup E_n$ is *not* σ finite. Then there is at least one n such that E_n is not σ finite; if otherwise, we have that $E_n = \bigsqcup K_{i,n}$ for all n, $\mu(K_{i,n}) < \infty$ for every i, and so $\bigsqcup E_n = \bigsqcup \bigsqcup K_{n,i} = \bigsqcup K_j$, $\mu(K_j) < \infty$, giving us a contradiction. Since there is at least one, we get the following equality:

$$\infty = \rho\left(\bigsqcup E_n\right) = \sum \rho(E_n) = \infty.$$

So, we have that ρ is indeed a measure. Next, we check that for all $E \in \mathcal{M}$, $\mu(E) = \rho(E) + \nu(E)$. We do this by breaking it up into cases.

Case 1: Assume $\mu(E) < \infty$. Then this clearly follows, since $\nu(E) = \mu(E)$, E is σ-finite and so $\rho(E) = 0$.

Case 2: Assume $\mu(E) = \infty$. Furthermore, assume E is σ-finite. Then we have that $E = \bigsqcup F_i$, $\mu(F_i) < \infty$. Define $G_n = \bigcup_{i=1}^n F_i$. Then we have $\nu(G_n) \le \nu(E)$ for all n, and so therefore $\nu(E) = \infty$, $\rho(E) = 0$.

Case 3: Assume $\mu(E) = \infty$. Furthermore, assume E is not σ-finite. Then we have that $\rho(E) = \infty$, and we're done.

Thus, we have that $\mu = \rho + \nu$.

□

Problem 8. Suppose $\mathcal{A}$ is an algebra on X, and let $\mathcal{M}$ be the σ-algebra generated by $\mathcal{A}$. Let μ_0 be a σ-finite premeasure on $\mathcal{A}$, μ^* the induced outer measure, and $\mathcal{M}^*$ the σ-algebra of μ^*-measurable sets. Show that the following are equivalent:

(1) $E \in \mathcal{M}^*$;
(2) $E = F - N$, where $F \in \mathcal{M}$ and $\mu^*(N) = 0$;
(3) $E = F \cup N$, where $F \in \mathcal{M}$ and $\mu^*(N) = 0$.

Deduce that μ is a σ-finite measure on $\mathcal{M}$, then $\mu^*|_{\mathcal{M}^*}$ on M^* is the completion of μ on $\mathcal{M}$.

Proof. Recall that μ_0 induces an outer measure via

$$\mu^*(E) = \inf\left\{\sum_1^\infty \mu_0(A_j) \;:\; A_j \in \mathcal{A},\; E \subset \bigcup_1^\infty A_j\right\}.$$

We proceed as how Folland does it. We start with a claim.

Claim. For any $E \subset X$ and $\epsilon > 0$, there exists $A \in \mathcal{A}_\sigma$ with $E \subset A$ and $\mu^*(A) \le \mu^*(E) + \epsilon$.

Proof. Fix $E \subset X$, $\epsilon > 0$. Since $\mu^*(E)$ is an infimum, we have that there must be a cover $A = \bigcup_j A_j$, where $A \in \mathcal{A}_\sigma$ since it is a countable union, of E so that

$$\mu^*(A) = \mu^*\left(\bigcup_j A_j\right) \le \sum_j \mu^*(A_j) = \sum_j \mu_0(A_j) \le \mu^*(E) + \epsilon.$$

Notice here we used the fact that, on $A \in \mathcal{A}$, we have $\mu^*(A) = \mu_0(A)$. Thus, since the choice of E and $\epsilon > 0$ were arbitrary, we get that for any $E \subset X$ and $\epsilon > 0$, there is an $A \in \mathcal{A}_\sigma$ so that $\mu^*(A) \le \mu^*(E) + \epsilon$. □

(1) $\implies$ (2): Since μ_0 is σ-finite, we have that there is a cover

$$\bigcup_i K_i = X$$

such that $\mu_0(K_i) = \mu^*(K_i) < \infty$ for all i. Let $E \in \mathcal{M}^*$ and write $E_n = E \cap K_n$. Then we have $\mu^*(E_n) \leq \mu^*(K_n) < \infty$ by monotonicity. Fix an $\epsilon > 0$. By the claim above, we can find $\{F_n\} \subset \mathcal{M}$, $E \subset F_n$ for every n, so that

$$\mu^*(F_n) \leq \mu^*(E_n) + \frac{\epsilon}{2^n}.$$

Since $\mu^*(E_n), \mu^*(F_n) < \infty$, and E_n, F_n are measurable, we can subtract $\mu^*(E_n)$ from both sides and rewrite it as

$$\mu^*(F_n) - \mu^*(E_n) = \mu^*(F_n - E_n) \leq \frac{\epsilon}{2^n}.$$

Now, letting $F = \bigcup F_n$, we have

$$F - E \subset \bigcup (F_n - E_n),$$

and so using subadditivity of measures we get

$$\mu^*(F - E) \leq \sum \mu^*(F_n - E_n) \leq \epsilon.$$

Taking $\epsilon = 1/k$, let F_k be the set such that $E \subset F_k$ and

$$\mu^*(F_k - E) \leq \frac{1}{k}.$$

If we let $F = \bigcap_k F_k$, we see that we have an F such that for all $\epsilon > 0$,

$$\mu^*(F - E) < \epsilon.$$

In other words, $\mu^*(F - E) = 0$. Notice that we can write E now as

$$E = F - (F - E),$$

where $\mu^*(F - E) = 0$ and $F \in \mathcal{M}$, as desired.

(2) $\implies$ (1): We have $E = F - N$, $F \in \mathcal{M}$ and $\mu^*(N) = 0$. We want to show that for all $G \subset X$, we have

$$\mu^*(G) = \mu^*(G \cap E) + \mu^*(G \cap E^C) = \mu^*(G \cap F \cap N^C) + \mu^*(G \cap (F^C \cup N)).$$

Notice ahead of time that

$$\mu^*(G) \leq \mu^*(G \cap E) + \mu^*(G \cap E^C)$$

by subadditivity, so it suffices to show that

$$\mu^*(G \cap E) + \mu^*(G \cap E^C) \leq \mu^*(G).$$

Rewrite this as

$$\mu^*(G \cap F \cap N^C) + \mu^*(G \cap (F^C \cup N)) = \mu^*(G \cap F \cap N^C) + \mu^*((G \cap F^C) \cup (G \cap N)).$$

Subadditivity then gives

$$\mu^*(G \cap F \cap N^C) + \mu^*((G \cap F^C) \cup (G \cap N)) \leq \mu^*(G \cap F \cap N^C) + \mu^*(G \cap F^C) + \mu^*(G \cap N).$$

Monotonicity gives

$$\mu^*(G \cap N) \leq \mu^*(N) = 0,$$

so we get

$$\mu^*(G \cap E) + \mu^*(G \cap E^C) \leq \mu^*(G \cap F \cap N^C) + \mu^*(G \cap F^C).$$

We use monotonicity again, noticing that $G \cap F \cap N^C \subset G \cap F$, so

$$\mu^*(G \cap F \cap N^C) \leq \mu^*(G \cap F).$$

Hence, we have

$$\mu^*(G \cap E) + \mu^*(G \cap E^C) \le \mu^*(G \cap F) + \mu^*(G \cap F^C) = \mu^*(G),$$

since $F \in \mathcal{M}$ and μ^* is a measure on $\mathcal{M}$ by **Theorem 1.14**. So we have that $E \in \mathcal{M}^*$.

So we have established (1) $\iff$ (2).

(1) $\implies$ (3) : Since $E \in \mathcal{M}^*$, $\mathcal{M}^*$ is a σ-algebra, we have $E^C \in \mathcal{M}^*$. By (1) $\iff$ (2), we can write $E^C = F - N = F \cap N^C$, $F \in \mathcal{M}$. Let $G^C = F$, then $G^C \in \mathcal{M}$ since it is a σ-algebra and we have $E^C = G^C \cap N^C$. Taking the complement of both sides gives us $E = G \cup N$, where $G \in \mathcal{M}$ and $\mu^*(N) = 0$.

(3) $\implies$ (1) We have $E = F \cup N$, $F \in \mathcal{M}$ and $\mu^*(N) = 0$. We want to show that $E \in \mathcal{M}^*$. That is, for all $G \subset X$, we have

$$\mu^*(G) = \mu^*(G \cap E) + \mu^*(G \cap E^C).$$

Subadditivity gives us

$$\mu^*(G) \le \mu^*(G \cap E) + \mu^*(G \cap E^C),$$

so it suffices to show the other direction. Notice that

$$\mu^*(G \cap E) + \mu^*(G \cap E^C) = \mu^*(G \cap (F \cup N)) + \mu^*(G \cap (F^C \cap N^C)).$$

Letting $H = F^C$, we have that $H \in \mathcal{M}$ and

$$\mu^*(G \cap E) + \mu^*(G \cap E^C) = \mu^*(G \cap (H^C \cup N)) + \mu^*(G \cap (H \cap N^C)),$$

and we see that this is the same scenario as the argument in (2) $\implies$ (1). Hence, we have

$$\mu^*(G \cap E) + \mu^*(G \cap E^C) \le \mu^*(G),$$

and so $E \in \mathcal{M}^*$. Thus, we have (1) $\iff$ (2), and (1) $\iff$ (3), so we can deduce (2) $\iff$ (3). Thus, (1), (2), and (3) are equivalent.

From the above equivalence, we have that $\mathcal{M}^* = \{F \cup N \ : \ F \in \mathcal{M} \text{ and } \mu^*(N) = 0\}$. Hence, $\mathcal{M}^*$ is the completion of $\mathcal{M}$, and furthermore μ^* is the completion of μ. ☐

Problem 9. Let μ^* be an outer measure on $P(X)$, $\mathcal{M}^*$ the σ-algebra of μ^*-measurable sets, and $\mu := \mu^*|_{\mathcal{M}^*}$. Let μ^+ be the outer measure on $P(X)$ induced by the (pre)measure μ on the (σ-)algebra $\mathcal{M}^*$.

(1) Show that $\mu^*(E) \le \mu^+(E)$ for all $E \subset X$, with equality if and only if there is an $F \in \mathcal{M}^*$ with $E \subset F$ and $\mu^*(E) = \mu^*(F)$.
(2) Show that if μ^* was induced from a premeasure μ_0 on an algebra $\mathcal{A}$, then $\mu^* = \mu^+$.
(3) Construct an outer measure μ^* on the two point set $X = \{0, 1\}$ such that $\mu^* \ne \mu^+$.

Proof. (1) We have

$$\mu^+(E) = \inf\left\{\sum_1^\infty \mu(A_j) \ : \ A_j \in \mathcal{M}^*, E \subset \bigcup_1^\infty A_j\right\}.$$

We have $\mu^*(E) \le \mu^+(E)$ since for all such covers of E we have

$$\mu^*(E) \le \mu^*\left(\bigcup_i A_i\right) \le \sum_i \mu^*(A_i) = \sum_i \mu(A_i),$$

by monotonicity and subadditivity, and the fact that the A_i are μ^* measurable. Thus, $\mu^*(E)$ is a lower bound of $\sum \mu(A_j)$ where the $A_j \in \mathcal{M}^*$ and they form a cover of E. Since $\mu^+(E)$ is the infimum, we have it is the greatest lower bound over all such covers. Thus $\mu^*(E) \le \mu^+(E)$.

Assume that $\mu^*(E) = \mu^+(E)$. In the case that $\mu^*(E) = \infty$, take $F = X$. Otherwise, we have $\mu^*(E) < \infty$. So, for all $\epsilon > 0$, we can find a cover $\{F_i\} \subset \mathcal{M}^*$ so that

$$\mu^+(E) \le \sum \mu(F_i) < \mu^*(E) + \epsilon.$$

Noticing that $\mu(F_i) = \mu^*(F_i)$ for $F_i \in \mathcal{M}^*$, $\mu^+(E) = \mu^*(E)$, and $\mu^*(E) \le \mu^*\left(\bigcup_i F_i\right) \le \sum \mu^*(F_i)$, we can write this as

$$\mu^*(E) \le \mu^*\left(\bigcup F_i\right) \le \sum \mu^*(F_i) < \mu^*(E) + \epsilon.$$

Taking $\epsilon = 1/k$ and the corresponding $\bigcup_i F_i$ as $\bigcup_i F_{i,k}$, we have that

$$\mu^*(E) \le \mu^*\left(\bigcup_i F_{i,k}\right) < \mu^*(E) + 1/k.$$

Let $F = \bigcap_k \bigcup_i F_{i,k}$. Then $F \subset \bigcup_i F_{i,k}$ for every k, and $E \subset F$ still, so we have

$$\mu^*(E) \le \mu^*(F) \le \mu^*\left(\bigcap_k \bigcup_i F_{i,k}\right) < \mu^*(E) + 1/k.$$

Since this applies for every k, we have that for all $\epsilon > 0$,

$$\mu^*(E) \le \mu^*(F) < \mu^*(E) + 1/k,$$

and so therefore $\mu^*(F) = \mu^*(E)$. Since $\mathcal{M}^*$ is closed under countable intersections and unions, we get that $F \in \mathcal{M}^*$.

Going the other direction, we have that $\mu^*(E) \le \mu^+(E)$, and we assume there is an $F \in \mathcal{M}^*$ with $E \subset F$ and $\mu^*(E) = \mu^*(F)$. We would like to show that $\mu^+(E) \le \mu^*(E)$. To do so, take the cover $E \subset F \cup \bigcup_2^\infty \varnothing$. Then we have that

$$\mu^+(E) \le \mu(F) + \sum_2^\infty \mu(\varnothing) = \mu(F) = \mu^*(F) = \mu^*(E).$$

Hence, $\mu^+(E) = \mu^*(E)$.

(2) If μ^* is induced from a premeasure μ_0, then we have

$$\mu^* = \inf\left\{\sum_1^\infty \mu_0(A_j) \ : \ A_j \in \mathcal{A},\ E \subset \bigcup_1^\infty A_j\right\}.$$

We would like to show that, for arbitrary $E \subset X$, $\mu^*(E) = \mu^+(E)$. From (1), we see that it suffices to show that there is an $F \in \mathcal{M}^*$ so that $E \subset F$ and $\mu^*(E) = \mu^*(F)$. In the case that $\mu^*(E) = \infty$, notice that we can take $X \in \mathcal{M}^*$. Then $E \subset X$ by assumption, $\mu^*(E) \le \mu^*(X)$, and so $\mu^*(E) = \mu^*(X)$. Therefore, $\mu^+(E) = \mu^*(E)$.

Assume now that $\mu^*(E) < \infty$. From the claim in **Problem 3**, we have that we can find $F \in A_\sigma \subset \mathcal{M}^*$ so that $\mu^*(F) \le \mu^*(E) + \epsilon$. Take $\epsilon = 1/k$, F_k the corresponding set so that

$$\mu^*(F_k) \le \mu^*(E) + 1/k,$$

and let $F = \bigcap F_k \in \mathcal{M}^*$ to get that $\mu^*(E) = \mu^*(F)$. Thus, $\mu^+(E) = \mu^*(E)$.

(3) Notice that $P(X) = \{\varnothing, \{0\}, \{1\}, X\}$. Then let $\mu^*(\varnothing) = 0$, $\mu^*(0) = 2, \mu^*(1) = 2$, $\mu^*(X) = 3$. We have that μ^* is subadditive, monotone, and $\mu^*(\varnothing) = 0$, so it is an outer measure. Recall that $\mathcal{M}^*$ is the collection of all μ^* measurable sets, meaning the sets A such that for all $E \subset X$, we have $\mu^*(E) = \mu^*(A \cap E) + \mu^*(A^C \cap E)$. We have $X, \varnothing \in \mathcal{M}$, but we see that $\{1\}$ and $\{0\}$ are not in $\mathcal{M}$; this is since

$$3 = \mu^*(X) \ne \mu^*(X \cap \{1\}) + \mu^*(X \cap \{0\}) = \mu^*(\{1\}) + \mu^*(\{0\}) = 4.$$

By definition, we see that

$$\mu^+(\{1\}) = 3 \neq 2 = \mu^*(\{1\}),$$

so $\mu^* \neq \mu^+$.

□

Problem 10. Suppose μ_0 is a finite premeasure on the algebra $A \subset P(X)$, and let $\mu^* : P(X) \to [0, \infty]$ be the outer measure induced by μ_0. Prove that the following are equivalent for $E \subset X$:

(1) $E \in \mathcal{M}^*$, the μ^*-measurable sets;

(2) $\mu^*(E) + \mu^*(E^C) = \mu^*(X)$.

Proof. Throughout, let $\mathcal{M} = \mathcal{M}(A)$; the σ-algebra generated by A.

(1) $\implies$ (2) : The definition of $E \in \mathcal{M}^*$ says that for all $A \subset X$, we have

$$\mu^*(A) = \mu^*(E \cap A) + \mu^*(E^C \cap A).$$

Taking $A = X$ gives us (2).

(2) $\implies$ (1) : We want to show that we can write $E = F - N$ for F which is $\mathcal{M}^*$ measurable and $\mu^*(N) = 0$. We see that the claim from **Problem 3** gives us that there is a set $F \in \mathcal{M}$, $E \subset F$ so that $\mu^*(F) \leq \mu^*(E) + \epsilon$, and since μ_0 is a finite measure, we get $\mu^*(F) - \mu^*(E) \leq \epsilon$. Since E is not necessarily measurable, we can no longer deduce that $\mu^*(F) - \mu^*(E) = \mu^*(F - E)$, so we need to proceed a different route.

Choosing $\epsilon = 1/n$ and the respective F which satisfies this as F_n, we get $\bigcap_n F_n = G \in \mathcal{M}$ is such that $\mu^*(G) - \mu^*(E) = 0$; that is, $\mu^*(G) = \mu^*(E)$. Since G is μ^* measurable, we have

$$\mu^*(E) + \mu^*(E^C) = \mu^*(X) = \mu^*(G) + \mu^*(G^C).$$

Since $\mu^*(E) = \mu^*(G)$ and $\mu^*(X) < \infty$, this gives us $\mu^*(G^C) = \mu^*(E^C)$. The measurability of G then tells us that

$$\mu^*(E^C) = \mu^*(G \cap E^C) + \mu^*(G^C \cap E^C),$$

so that

$$\mu^*(G^C) = \mu^*(G \cap E^C) + \mu^*(G^C \cap E^C).$$

Since $E \subset G$, we have $G^C \subset E^C$ so that $G^C \cap E^C = G^C$. Hence, we may rewrite this as

$$\mu^*(G^C) = \mu^*(G \cap E^C) + \mu^*(G^C) \leftrightarrow \mu^*(G - E) = 0.$$

Hence, write $E = G - (G - E)$ to get that $E \in \mathcal{M}^*$ by **Problem 3**. □

Problem 11. (1) Show that every open subset of $\mathbb{R}$ is a countable union of open intervals where both endpoints are rational.

(2) Suppose $U \subset \mathbb{R}$ is open and suppose $((a_j, b_j))_{j\in J}$ is a collection of open intervals which cover U:

$$U \subset \bigcup_{j\in J}(a_j, b_j)$$

Show that there is a countable sub-cover, i.e., show that there is a countable subset $I \subset J$ such that

$$U \subset \bigcup_{i\in I}(a_i, b_i).$$

(3) Suppose $((a_i, b_j])_{j\in J}$ is a collection of half-open intervals which cover $(0, 1]$:

$$(0, 1] \subset \bigcup_{j\in J}(a_j, b_j].$$

Show that there is a countable sub-cover, i.e., show that there is a countable subset $I \subset J$ such that

$$(0, 1] \subset \bigcup_{i\in I}(a_i, b_i].$$

Proof. (1) We first establish it for open balls, (a, b) where $-\infty < a < b < \infty$. Since the rationals are dense, we have a decreasing sequence $a_n \in (a, b)$, a_n rational, such that $a_n \to a$ as $n \to \infty$. Likewise, we have that there is an increasing sequence $b_n \in (a, b)$, such that $b_n \to b$ as $n \to \infty$ and the b_n are rational. Thus, we need to show that we can write this as

$$(a, b) = \bigcup_{n=1}^{\infty}(a_n, b_n).$$

Notice that

$$\bigcup_{n=1}^{\infty}(a_n, b_n) \subset (a, b).$$

For the other direction, take $x \in (a, b)$. Then we have either $x < a_1$, $a_1 \le x \le b_1$, or $x > b_1$. If $a_1 \le x \le b_1$, we get that $x \in \bigcup_{n=1}^{\infty}(a_n, b_n)$ and we're done. Without loss of generality, assume that $x < a_1$. Since $a_n \to a$, and $x > a$, we have that there must be some n such that $x > a_n$; if otherwise (that is, for all n, $x \le a_n$), then we notice that for all $\epsilon > 0$ there is an N such that for all $n \ge N$, we have $a_N - a < \epsilon$. Thus, we have for all $\epsilon > 0$, $a < x \le a + \epsilon$, which forces $x = a$, a contradiction to the choice of x. Thus, there is some n such that $x > a_n$, which says that $x \in (a_n, b_n)$, and therefore $x \in \bigcup_{n=1}^{\infty}(a_n, b_n)$. The argument for $x > b_1$ is analogous, and so we get for all $x \in (a, b)$, $x \in \bigcup_{n=1}^{\infty}(a_n, b_n)$. Hence,

$$(a, b) = \bigcup_{n=1}^{\infty}(a_n, b_n).$$

Now, we've established for open balls, and we notice that open balls form a basis for the topology of $\mathbb{R}$. So, if $U \subset \mathbb{R}$ is an open set, then we can write it as

$$U = \bigcup_{n=1}^{\infty}(a_n, b_n),$$

and from our prior step we can write

$$(a_n, b_n) = \bigcup_{j=1}^{\infty} (a_{n,j}, b_{n,j}),$$

where the $a_{n,j}, b_{n,j}$ are rational numbers. Hence, we can write

$$U = \bigcup_{n,j=1}^{\infty} (a_{n,j}, b_{n,j}),$$

or after reordering indices that

$$U = \bigcup_{k=1}^{\infty} (a_k, b_k),$$

where the a_k, b_k are rational numbers. This is a countable union.

(2) From (1), we can write U open as

$$U = \bigcup_{k=1}^{\infty} (p_k, q_k)$$

where the p_k, q_k are rational numbers. Now, let $I = \{i \in \mathbb{N} : (p_i, q_i) \subset (a_j, b_j) \text{ for some } j \in J\}$. This is clearly countable, since it is a subset of a countable set. For each $i \in I$, take (a_i, b_i) so that $(p_i, q_i) \subset (a_i, b_i)$. Then we see that

$$U \subset \bigcup_{i \in I} (a_i, b_i),$$

since the (p_k, q_k) cover U, and so we're done.

(3) Let

$$U = \bigcup_{j \in J} (a_j, b_j).$$

Then U is an open set, and so by (2) we can refine it so that

$$U = \bigcup_{i \in I} (a_i, b_i),$$

where I is countable. Now, let's examine $Y := (0, 1] - U$. Take $x \in Y$. Notice that $x = b_j$ for some $j \in J$; if otherwise, we have that it must be in (a_j, b_j) for some $j \in J$, since these cover $(0, 1]$, but then this would be in U, a contradiction. So we have that all of the points that we missed are endpoints.

To get that there are countably many, take some rational number $q_x \in (a_j, b_j)$, where j is such that $x = b_j$. The existence of such a rational number comes from the density of rationals. Notice that $(q_x, x) \subset U$. Construct a function $f : Y \to \mathbb{Q}$ such that $f(x) = q_x$, where q_x is the associated rational number chosen in the interval given above. If we can show that f is well-defined and injective, we get that Y is countable, and so we can throw in the intervals of the form $\{(a_j, b_j] : b_j \in Y\}$ into U to get a countable cover of $(0, 1]$.

To see that f is well-defined, notice that $x = y$ implies that $f(x) = q_x = q_y = f(y)$ so long as we are consistent with our choice of rational. Assuming that we are, we get that this function is well-defined.

To see that f is injective, we check if $f(x) = f(y)$, then $x = y$. We do this by contrapositive; that is, if $x \neq y$, then $f(x) \neq f(y)$. To do so, assume that $x < y$, but $q_y \leq q_x$. Then we have $q_y \leq q_x < x < y$. But this implies that $x \in (q_y, y)$ which is in U; a contradiction to the fact that we chose $x, y \in Y$. Hence, we must have that $q_y > q_x$; that is, if $x \neq y$, then $f(x) \neq f(y)$.

Since the function is injective and well-defined, we get that Y must be countable. Hence, by the argument given from prior, we can form $I' := I \cup \{j \in J \ : \ b_j \in Y\}$ to be a countable set, and we have

$$(0,1] \subset \bigcup_{i \in I'} (a_i, b_i].$$

□

Problem 12. Define the h-intervals

$$\mathcal{H} = \{\varnothing\} \cup \{(a,b] \ : \ -\infty \le a < b < \infty\} \cup \{(a,\infty) \ : \ a \in \mathbb{R}\}.$$

Let $\mathcal{A}$ be the collection of finite disjoint unions of elements in $\mathcal{H}$. Show directly from the definitions that $\mathcal{A}$ is an algebra. Deduce that the σ-algebra $\mathcal{M}(A)$ generated by $\mathcal{A}$ is equal to the Borel σ-algebra $\mathcal{B}_{\mathbb{R}}$.

Proof. We proceed from the definition. We need to show three things:

- $\mathbb{R} \in \mathcal{A}$ (not necessary, but useful for future parts);
- If $A \in \mathcal{A}$, $A^C \in \mathcal{A}$;
- If $\{A_i\}_{i=1}^n \subset \mathcal{A}$, then $\bigcup_{i=1}^n A_i \in \mathcal{A}$.

We first show that $\mathbb{R} \in \mathcal{A}$. Take some point $b \in \mathbb{R}$. Then we have $(-\infty, b] \in \mathcal{H}$, $(b,\infty) \in \mathcal{H}$, $(-\infty,b] \cap (b,\infty) = \varnothing$, so $(-\infty,b] \sqcup (b,\infty) = \mathbb{R} \in \mathcal{H}$, since for all $x \in \mathbb{R}$ we have either $x \le b$ or $x > b$.

We now show complements. Take $A \in \mathcal{A}$. Then we can write A as

$$A = \bigsqcup_{i=1}^n F_i,$$

where the $F_i \in \mathcal{H}$. Notice then that

$$A^C = \left(\bigsqcup_{i=1}^n F_i \right) = \bigcap_{i=1}^n F_i^c.$$

We have that the F_i could be of three forms; either $F_i = \varnothing$, $F_i = (a,b]$, where $-\infty \le a < b < \infty$, or $F_i = (a,\infty)$, where $a \in \mathbb{R}$. If $F_i = \varnothing$, then $F_i^C = \mathbb{R}$, if $F_i = (a,b]$, then $F_i^C = (-\infty,a] \sqcup (b,\infty)$, and if $F_i = (a,\infty)$ then $F_i^C = (-\infty,a]$. In all of the above cases, we still have that the F_i are at most finite disjoint unions of $\mathcal{H}$ intervals. Notice that if we have an F_i^C of the first form intersected with any of the other two, we just get a finite disjoint union of $\mathcal{H}$ intervals. If we have an F_i^C of the second form (say $F_i^C = (-\infty,a'] \sqcup (b,\infty)$) intersected with an F_j^C of the third form (say $F_j^C = (-\infty,a]$), then there are three things that can happen: if $a < a'$, then $F_i^C \cap F_j^C = (-\infty,a]$, if $a' \le a \le b$, we have $F_i^C \cap F_j^C = (-\infty,a']$, and if we have $a > b$, then we get $F_i^C \cap F_j^C = (-\infty,a'] \sqcup (b,a]$. In all of the cases above, we see we get at most a finite disjoint union of $\mathcal{H}$ intervals, and so the intersection under this complement will be at most a finite disjoint union of $\mathcal{H}$ intervals; in other words, $A^C \in \mathcal{A}$.

Finally, we show it's closed under finite unions. It suffices to show that if $A, B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$. Write

$$A = \bigsqcup_{i=1}^n F_i,$$

$$B = \bigsqcup_{j=1}^m G_j,$$

where F_i, G_i is of one of the forms given above. We can then write

$$A \cup B = \bigsqcup_{i=1}^{n} F_i \cup \bigsqcup_{j=1}^{m} G_j.$$

If the F_i, G_j are disjoint, then we are done. Otherwise, we have that there is an i, j so that $F_i \cap G_j \neq \varnothing$. We iterate through the three cases (since $F_i = \varnothing$ or $G_j = \varnothing$ cannot result in this): If $F_i = (a, b]$, $G_j = (c, d]$, then $F_i \cap G_j \neq \varnothing$ implies that either $G_j \subset F_i$, $F_i \subset G_j$, $a < c < b < d$, or $c < a < d < b$. If $F_i \subset G_j$, then $G_j \cup F_i = G_j$, and so we are done. Assume without loss of generality that $a < c < b < d$. Then $F_i \cup G_j = (a, b] \cup (c, d] = (a, d]$, which is still an $\mathcal{H}$ interval. So, in this case, we have $F_i \cup G_j$ is just a single $\mathcal{H}$ interval.

If $F_i = (a, \infty)$, $G_j = (b, c]$, then we see that $F_i \cap G_j \neq \varnothing$ can be due to two things; either $G_j \subset F_i$ or $b < a < c$. In the first case, we see that $F_i \cup G_j = F_i$, and so we get a single $\mathcal{H}$ interval. In the second case, we see that $F_i \cup G_J = (a, \infty) \cup (b, c] = (b, \infty)$, which is still a single $\mathcal{H}$ interval. Thus, we see that the union here is a single $\mathcal{H}$ interval.

Finally, we consider the case where $F_i = (a, \infty)$, $G_j = (b, \infty)$. If $F_i \cap G_j \neq \varnothing$, then we either have $a < b$, $b < a$, or $b = a$. If $a < b$, we get that $F_i \cup G_j = (a, \infty) \cup (b, \infty) = (a, \infty)$, and an analogous result for if $b < a$. If $b = a$, then $F_i \cup G_j = F_i$. So we see that we get that the union here is a single $\mathcal{H}$ interval, like before.

So, iterating this, we see that we can union all of the non-disjoint intervals F_i, G_j to get a disjoint union of $\mathcal{H}$ intervals. Thus, $A \cup B \in \mathcal{H}$.

To see why it suffices to show it for two $\mathcal{A}$ elements, we proceed by induction on the number of $\mathcal{A}$ elements. It clearly holds for $n = 1, 2$ from above work, so assume it holds for $n - 1$. We want to then show it holds for n. We have

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n-1} A_i \cup A_n = B \cup A_n \in \mathcal{A},$$

since $\bigcup_{i=1}^{n-1} A_i \in \mathcal{A}$ by assumption, we write it as $B \in \mathcal{A}$, and from the $n = 2$ case we see that $B \cup A_n \in \mathcal{A}$. Thus, it holds for all finite unions.

Since we have the three properties, we get that $\mathcal{A}$ is an algebra. Furthermore, notice that we can write every open ball as a countable union of elements in $\mathcal{H}$; that is, if we have some open ball (a, b), $-\infty \leq a < b < \infty$, we can write it as the countable union

$$\bigcup_{n=1}^{\infty} (a, b + 1/n].$$

If we have an open ball (a, ∞), we see this in $\mathcal{H}$ by assumption. Thus, we get all possible open balls from this construction. Since the open balls form a basis for the topology of $\mathbb{R}$, we can write open sets in $\mathbb{R}$ as countable unions of open balls, and so as countable unions of elements from $\mathcal{H}$. Thus, $\mathcal{B}_{\mathbb{R}} \subset \mathcal{M}(\mathcal{A})$ by the minimality of $\mathcal{B}_{\mathbb{R}}$, since we have all of the open sets. For the other direction, we clearly note that $\mathcal{A} \subset \mathcal{B}_{\mathbb{R}}$, since intersecting open and closed sets will give us $\mathcal{H}$ intervals and we can use the σ-algebra property of $\mathcal{B}_{\mathbb{R}}$ to get all of the disjoint unions. Thus, by the minimality of $\mathcal{M}(\mathcal{A})$, we have $\mathcal{M}(\mathcal{A}) \subset \mathcal{B}_{\mathbb{R}}$, which implies they are equal. □

Problem 13. Assume the notation of the prior problem. Suppose $F : \mathbb{R} \to \mathbb{R}$ is non-decreasing and right continuous. Extend F to a function $[-\infty, \infty] \to [-\infty, \infty]$, still denoted F, by

$$F(-\infty) := \lim_{a \to -\infty} F(a), \quad F(\infty) := \lim_{b \to \infty} F(b).$$

Define $\mu_0 : \mathcal{H} \to [0, \infty]$ by

- $\mu_0(\varnothing) = 0$,

- $\mu_0((a,b]) := F(b) - F(a)$ for all $-\infty \le a < b < \infty$, and
- $\mu_0((a,\infty)) := F(\infty) - F(a)$ for all $a \in \mathbb{R}$.

Suppose $(a,\infty) = \bigsqcup_{j=1}^{\infty} H_j$, where $(H_j) \subset \mathcal{H}$ is a sequence of disjoint h-intervals. Show that

$$\mu_0((a,\infty)) = \sum_{j=1}^{\infty} \mu_0(H_j).$$

Proof. One direction follows from the class notes: we have for $n \in \mathbb{N}$ fixed that

$$\bigsqcup_{j=1}^{n} H_j \subset (a,\infty),$$

and monotonicity and finite additvity gives gives

$$\mu_0\left(\bigsqcup_{j=1}^{n} H_j\right) = \sum_{j=1}^{n} \mu_0(H_j) \le \mu_0((a,\infty)).$$

This applies for all $n \in \mathbb{N}$, so we can take the limit to get

$$\sum_{j=1}^{\infty} \mu_0(H_j) \le \mu_0((a,\infty)).$$

It suffices then to prove the other direction. Notice that, from our work in the prior problem, if we have a disjoint union of H_j intervals, we must have it's of the form

$$(a_0,\infty) \sqcup \bigsqcup_{j=1}^{\infty} (a_j, b_j],$$

or

$$\bigsqcup_{j=1}^{\infty} (a_j, b_j],$$

where the $b_j \nearrow \infty$ as $j \to \infty$. The fact that we do not have multiple intervals of the form (a,∞) follows: for all $a, a' \in [-\infty,\infty)$, we have that $(a,\infty) \cap (a',\infty) \neq \varnothing$; this follows from the fact that either $a \le a'$ or $a' < a$. If $a \le a'$, then we have $(a,\infty) \subset (a',\infty)$, and so they're not disjoint, and if $a' < a$ we have a similar issue.

We can convert sets of the first type to sets of the second type by simply observing that

$$(a_0,\infty) = \bigsqcup_{j=1}^{\infty} (a_j, b_j],$$

where $a_1 = a_0$, and the b_j are some sequence of rationals increasing to infinity, where $a_{j+1} = b_j$. So it suffices to study (H_j) of the second type in order to prove the statement.

We follow the trick from the notes, with a slight modification. Take M finite such that $M > a$. Then we have $(a, M] \subset (a,\infty) = \bigsqcup_{j=1}^{\infty} H_j$. So the H_j cover $(a, M]$. Take $\epsilon > 0$ fixed (but arbitrary), then since we assumed F is right continuous, we have there is a $\delta > 0$ such that $F(a+\delta) - F(a) < \epsilon/2$, and for every j, there is a δ_j so that $F(b_j + \delta_j) - F(b_j) < \epsilon/2^{j+1}$. Notice that

$$\{(a_j, b_j + \delta_j)\}_{j=1}^{\infty}$$

forms an open cover of $[a+\delta, M]$. Since this is compact, we can take a finite refinement; that is, we have

$$[a+\delta, M] \subset \bigcup_{j=1}^{N} (a_j, b_j + \delta_j).$$

Now, we get

$$\mu_0((a, M]) = F(M) - F(a).$$

Adding and subtracting $F(a+\delta)$ gives

$$\mu_0((a, M]) = F(M) - F(a+\delta) + F(a+\delta) - F(a).$$

We chose δ so that $F(a+\delta) - F(a) < \epsilon/2$, so we substitute this in to get

$$\mu_0((a, M]) < F(M) - F(a+\delta) + \frac{\epsilon}{2}.$$

Now, we can use the fact that $\mu_0((a+\delta, M]) = F(M) - F(a+\delta)$ to rewrite this as

$$\mu_0((a, M]) < \mu_0((a+\delta, M]) + \frac{\epsilon}{2}.$$

We can now use the open cover, monotonicity, and finite subadditivity to rewrite this as

$$\mu_0((a, M]) < \mu_0((a+\delta, M]) + \frac{\epsilon}{2} \leq \mu_0\left(\bigcup_{j=1}^{N}(a_j, b_j+\delta_j])\right) + \frac{\epsilon}{2}$$

$$\leq \sum_{j=1}^{N} \mu_0((a_j, b_j+\delta_j]) + \frac{\epsilon}{2}.$$

We can rewrite this as

$$\mu_0((a, M]) < \sum_{j=1}^{N} [F(b_j+\delta_j) - F(a_j)] + \frac{\epsilon}{2}.$$

Add and subtract $F(b_j)$ inside the sum to get

$$\mu_0((a, M]) < \sum_{j=1}^{N} [F(b_j+\delta_j) - F(b_j) + F(b_j) - F(a_j)] + \frac{\epsilon}{2}.$$

Recall that we had $F(b_j+\delta_j) - F(b_j) < \epsilon/2^{j+1}$, so substituting this in gives

$$\mu_0((a, M]) < \sum_{j=1}^{N} \left[\frac{\epsilon}{2^{j+1}} + F(b_j) - F(a_j)\right] + \frac{\epsilon}{2}$$

$$= \sum_{j=1}^{N} [F(b_j) - F(a_j)] + \epsilon.$$

These are all positive values, since F is non-decreasing, so we can bound this above by

$$\mu_0((a, M]) < \sum_{j=1}^{N} [F(b_j) - F(a_j)] + \epsilon \leq \sum_{j=1}^{\infty} [F(b_j) - F(a_j)] + \epsilon$$

$$= \sum_{j=1}^{\infty} \mu_0((a_j, b_j]) + \epsilon.$$

Notice now that the left hand side can be written as

$$F(M) - F(a) < \sum_{j=1}^{\infty} \mu_0((a_j, b_j]) + \epsilon.$$

Since this works for arbitrary $M > a$, we can take the limit as $M \to \infty$ of both sides to get

$$F(\infty) - F(a) = \mu_0((a, \infty)) \leq \sum_{j=1}^{\infty} \mu_0((a_j, b_j]) + \epsilon.$$

Since this works for all $\epsilon > 0$, we get that

$$\mu_0((a, \infty)) \leq \sum_{j=1}^{\infty} \mu_0(H_j),$$

or in other words,

$$\mu_0((a, \infty)) = \sum_{j=1}^{\infty} \mu_0(H_j).$$

□

Problem 14 (Folland 1.28)**.** Let $F : \mathbb{R} \to \mathbb{R}$ be increasing and right continuous, and let μ_F be the associated Lebesgue-Stieltjes Borel measure on $\mathcal{B}_{\mathbb{R}}$. For $a \in \mathbb{R}$, define

$$F(a-) := \lim_{b \nearrow a} F(b).$$

Prove that

(1) $\mu_F(\{a\}) = F(a) - F(a-)$,
(2) $\mu_F([a, b]) = F(b) - F(a-)$,
(3) $\mu_F([a, b)) = F(b-) - F(a-)$, and
(4) $\mu_F((a, b)) = F(b-) - F(a)$.

Remark. I switched the order around to make things a little easier; what was (2) in the homework is (3) and vice versa.

Proof. (1) Notice that $\{a\} \subset (a - 1/n, a]$ for all $n \in \mathbb{N}$, and we have

$$\bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, a\right] = \{a\}.$$

Notice that $\mu_F(a - 1, a] = F(a) - F(a - 1) < \infty$. So, using continuity from above, we have

$$\mu_F(\{a\}) = \lim_{n \to \infty} \mu_0 \left(\left(a - \frac{1}{n}, a\right] \right) = \lim_{n \to \infty} F(a) - F\left(a - \frac{1}{n}\right) = F(a) - F(a-).$$

(2) Write $[a, b] = \{a\} \sqcup (a, b]$. Then we can use (1) to write this as

$$\mu_F([a, b]) = \mu_F(\{a\}) + \mu_F((a, b]) = F(a) - F(a-) + F(b) - F(a) = F(b) - F(a-).$$

(3) We break this up into cases.
Case 1: $(b < \infty)$ Notice that we can write this as

$$\bigcup_{n=1}^{\infty} \left[a, b - \frac{1}{n}\right] = [a, b).$$

Using (2) and continuity from below, we have

$$\mu_F([a, b)) = \lim_{n \to \infty} \mu_F([a, b - 1/n]) = \lim_{n \to \infty} F(b - 1/n) - F(a-) = F(b-) - F(a-).$$

Case 2: ($b = \infty$) Let $\{q_n\}_{n=1}^{\infty}$ be a countable sequence of increasing rational numbers greater than a which tend to infinity. Then we can write this as

$$\bigcup_{n=1}^{\infty} [a, q_n] = [a, b).$$

Using (2) and continuity from below, as well as the definition of the extension of F to $[-\infty, \infty]$, we have

$$\mu_F([a, b)] = \lim_{n\to\infty} \mu_F([a, q_n]) = \lim_{n\to\infty} F(q_n) - F(a-) = F(\infty) - F(a-) = F(b) - F(a-),$$

as desired.

(4) We again break this up into cases.
Case 1: ($b < \infty$) We can write this as

$$\bigcup_{n=1}^{\infty} \left(a, b - \frac{1}{n}\right] = (a, b).$$

Using continuity from below, we have

$$\mu_F((a, b)) = \lim_{n\to\infty} \mu_F((a, b - 1/n]) = F(b - 1/n) - F(a) = F(b-) - F(a).$$

Case 2: ($b = \infty$) Again, take a sequence of rational numbers increasing to ∞ which are greater than a (which may possibly be $-\infty$). We can write this as

$$\bigcup_{n=1}^{\infty} (a, q_n] = (a, b).$$

Using continuity from below and the definition, we get

$$\mu_F((a, b)) = \lim_{n\to\infty} \mu_F((a, q_n]) = \lim_{n\to\infty} F(q_n) - F(a) = F(b-) - F(a),$$

since we defined $F(b) = F(b-)$ in our extension of F.

□

Problem 15. Let (X, ρ) be a metric space. A subset $S \subset X$ is called nowhere dense if $\overline{S}$ does not contain any open set in X. A subset $T \subset X$ is called meager if it is a countable union of nowhere dense sets.

Construct a meager subset of $\mathbb{R}$ whose complement is Lebesgue null.

Proof. We proceed via the construction of a generalized Cantor set, found on page 39 of Folland. Let $\{\alpha_j\}_{j=0}^{\infty} \subset (0, 1)$ be a sequence of decreasing numbers; we can form a decreasing sequence $\{K_j\}$ of closed sets by taking $K_0 = [0, 1]$ and K_j is recursively defined by remove the open middle α_{j-1}th from each of the intervals that make up K_{j-1}. Setting $K = \bigcap_{j=1}^{\infty} K_j$, we get the generalized Cantor set. Notice that $\lambda(K_1) < \infty$, and so we can use continuity from above to deduce that

$$\lambda(K) = \lambda\left(\bigcap_{j=1}^{\infty} K_j\right) = \lim_{j\to\infty} \lambda(K_j).$$

Taking our α_j to be constant, say r, we see that

$$\lambda(K_j) = 1 - \sum_{k=0}^{j} 2^k r^{k+1} = 1 - \left(\frac{r - 2^j r^{j+1}}{1 - 2r}\right) > 0.$$

For appropriate choice of r, (that is, take it to be $r < 1/3$) we have

$$\lambda(K) = 1 - \left(\frac{r}{1-2r}\right).$$

Thus, fixing $\epsilon > 0$, for appropriate choice of r we can get that we can construct $K_\epsilon \subset [0,1]$ so that

$$\lambda(K_\epsilon) = 1 - \epsilon.$$

Taking $\epsilon = 1/n$, $n > 0$, we have

$$\lambda\left(K_{1/n}\right) = 1 - \frac{1}{n}.$$

Furthermore, we see that $K_{1/n}$ is nowhere dense; we note that $K_{1/n}$ is closed, since its a countable intersection of closed sets, and so it suffices to show that it has empty interior (thus saying that it does not contain any open set, since the interior is the largest open set contained in S). To see that it has empty interior, we go by contradiction. Take a point $x \in C$, then we can form an open ball of radius z centered at x, but we can find an p large enough so that $r^p < z$, which contradicts the existence of the open ball.

Now, we can let $G = \bigcup_{n=1}^{\infty} K_{1/n}$. Then G is meager, and we see that

$$K_{1/n} \subset G \subset [0,1]$$

for each n, and so

$$\lambda(K_{1/n}) \leq \lambda(G) \leq 1,$$

which implies that

$$1 - \frac{1}{n} \leq \lambda(G) \leq 1$$

for all $n > 1$. Thus, we have $\lambda(G) = 1$ and it is a meager set. Furthermore,

$$\lambda(G^c) + \lambda(G) = 1 \leftrightarrow \lambda(G^c) = 0.$$

We can now translate this set around; notice that

$$\mathbb{R} = \bigcup_{n=-\infty}^{\infty} G + n.$$

Call

$$X := \bigcup_{n=-\infty}^{\infty} G + n.$$

Then we see that X is a countable union of nowhere dense sets, since each G is, and so it is meager. Furthermore, we see that

$$\lambda\left(\mathbb{R} - X\right) \leq \sum_{n=-\infty}^{\infty} \lambda(G^c) = 0$$

using the translation invariance property of λ, and so we have a nowhere dense set X whose complement is Lebesgue null. $\square$

Remark. Thomas O'Hare was a collaborator for this homework set.

Problem 16. Suppose $E \in \mathcal{L}$ and $\lambda(E) > 0$.
(1) Show that for any $0 \leq \alpha < 1$, there is an open interval $I \subset \mathbb{R}$ such that $\lambda(E \cap I) > \alpha\lambda(I)$.
(2) Apply (1) with $\alpha = 3/4$ to show that the set

$$E - E = \{x - y \ : \ x, y \in E\}$$

contains the interval $(-\lambda(I)/2, \lambda(I)/2)$.

Proof. (1) **Step 1:** If $\alpha = 0$, then it suffices to find an open interval I so that $\lambda(E \cap I) > 0$. Assume for contradiction that, for every open interval I, we have $\lambda(E \cap I) = 0$. Cover $\mathbb{R}$ with disjoint open intervals, say

$$\mathbb{R} = \bigsqcup_{n=1}^{\infty} (p_n, q_n).$$

Then we have

$$E = E \cap \mathbb{R} = E \cap \bigsqcup_{n=1}^{\infty} (p_n, q_n) = \bigsqcup_{n=1}^{\infty} \left(E \cap (p_n, q_n)\right).$$

Hence,

$$\lambda(E) = \lambda\left(\bigsqcup_{n=1}^{\infty} (E \cap (p_n, q_n))\right) = \sum_{n=1}^{\infty} \lambda(E \cap (p_n, q_n)).$$

But since $\lambda(E \cap I) = 0$ for all open intervals, we have $\lambda(E \cap (p_n, q_n)) = 0$ for all n. Hence,

$$\lambda(E) = \sum_{n=1}^{\infty} 0 = 0.$$

But this is a contradiction, since we assumed $\lambda(E) > 0$. Hence, there must be some open interval I so that $\lambda(E \cap I) > 0$.

Step 2: Fix $0 < \alpha < 1$. Assume that $0 < \lambda(E) < \infty$. Then, for every $\epsilon > 0$, we can find an open set $E \subset U$ such that

$$\lambda(U) < \lambda(E) + \epsilon.$$

Since $0 < \alpha < 1$, we can write

$$\epsilon = \frac{1}{\alpha}\lambda(E) - \lambda(E) > 0,$$

and so there is an open set U so that

$$\lambda(U) < \lambda(E) + \frac{1}{\alpha}\lambda(E) - \lambda(E) = \frac{1}{\alpha}\lambda(E).$$

Notice that we can write U as a countable (disjoint) union of open intervals using properties of $\mathbb{R}$; that is,

$$U = \bigsqcup_{n=1}^{\infty} (a_n, b_n).$$

Step 3: Assume for contradiction that $\lambda(E \cap I) \leq \alpha\lambda(I)$ for all open intervals I. Since we chose $E \subset U$, we have that $E \cap U = E$. That is,

$$E = E \cap U = E \cap \bigsqcup_{n=1}^{\infty} (a_n, b_n) = \bigsqcup_{n=1}^{\infty} (E \cap (a_n, b_n)).$$

So,

$$\lambda(E) = \lambda\left(\bigsqcup_{n=1}^{\infty}(E \cap (a_n, b_n))\right) = \sum_{n=1}^{\infty} \lambda(E \cap (a_n, b_n)).$$

Now, by assumption, $\lambda(E \cap (a_n, b_n)) \leq \alpha\lambda((a_n, b_n))$, so we get

$$\lambda(E) \leq \sum_{n=1}^{\infty} \alpha\lambda((a_n, b_n)) = \alpha \sum_{n=1}^{\infty} \lambda((a_n, b_n)) = \alpha\lambda\left(\bigsqcup_{n=1}^{\infty}(a_n, b_n)\right) = \alpha\lambda(U).$$

But from **Step 2**, we had

$$\lambda(U) < \frac{1}{\alpha}\lambda(E) \leftrightarrow \alpha\lambda(U) < \lambda(E),$$

so we have

$$\lambda(E) < \lambda(E),$$

which is a contradiction. Hence, we must have $\lambda(E \cap I) > \alpha\lambda(I)$ for some open interval I.

Step 4: Assume now that $\lambda(E) = \infty$. Take a cover of $\mathbb{R}$ by bounded disjoint open balls; that is, write

$$\mathbb{R} = \bigsqcup_{n=1}^{\infty}(a_n, b_n).$$

Then we have

$$E = E \cap \mathbb{R} = E \cap \bigsqcup_{n=1}^{\infty}(a_n, b_n) = \bigsqcup_{n=1}^{\infty}(E \cap (a_n, b_n)).$$

Notice that

$$\lambda(E \cap (a_n, b_n)) < \infty,$$

and furthermore there is at least one n so that

$$0 < \lambda(E \cap (a_n, b_n)).$$

For such an n, we can use **Step 3** to get that for all $0 < \alpha < 1$, there is an interval I so that

$$\lambda((E \cap (a_n, b_n)) \cap I) > \alpha\lambda(I).$$

But montonicity gives

$$\lambda(E \cap I) \geq \lambda((E \cap (a_n, b_n)) \cap I) > \alpha\lambda(I),$$

and so we are done.

Thus, we've shown that for all E so that $\lambda(E) > 0$ and $E \in \mathcal{L}$, and for all $\alpha \in [0, 1)$, we can find an open interval I so that $\lambda(E \cap I) > \alpha\lambda(I)$.

(2) Take $\alpha = 3/4$. Then we can find an open interval I so that

$$\lambda(E \cap I) > \frac{3}{4}\lambda(I).$$

Let $F := E \cap I \subset E$. Then we have $F - F \subset E - E$. If we can show that $(-1/2\lambda(I), 1/2\lambda(I)) \subset F - F$, then we are done. To do that, we need to show that if $|z_0| < (1/2)\lambda(I)$, then $z_0 \in F - F$.

We first establish a claim that $F \cap (F + x_0) \neq \varnothing$ implies $x_0 \in F - F$. If $F \cap (F + x_0) \neq \varnothing$, then we have $y \in F \cap (F + x_0)$; that is, $y = x$, where $x \in F$, and $y = z + x_0$, where $z \in F$. Then we have $x = z + x_0$, or $x - z = x_0$, and so $x_0 \in F - F$.

Since $F \cap (F + 0) = F \cap F = F \neq \varnothing$, since $\lambda(F) > \alpha\lambda(I) > 0$, we get that $0 \in F$. Let z be such that $|z| < (1/2)\lambda(I)$. Notice that

$$\lambda(F) + \lambda(F + z) = \lambda(F \cap (F + z)) + \lambda(F \cup (F + z))$$

from **Quiz 1**. Since λ is translation invariant, we can bound this below by

$$\frac{3}{2}\lambda(I) \leq \lambda(F \cap (F+z)) + \lambda(F \cup (F+z)).$$

By monotonicity, we can see that

$$\lambda(F \cup (F+z)) \leq \lambda(I + (I+z)).$$

Now, assume $|z| = \frac{1}{2}\lambda(I)$. Then writing $I = (a, b)$, since it's an open interval, we can see that

$$I \cup (I+z) = (a, b+z),$$

and so

$$\lambda(I \cup (I+z)) = b + z - a = \frac{3}{2}\lambda(I).$$

Since this is a strict upper bound, we get

$$\lambda(F \cap (F+z)) > 0,$$

and so it's non-trivial for $z \in (-(1/2)\lambda(I), (1/2)\lambda(I))$. Hence,

$$\left(-\frac{1}{2}\lambda(I), \frac{1}{2}\lambda(I)\right) \subset F - F \subset E - E,$$

as desired.

□

Problem 17. Let $\mathcal{B}_{\mathbb{R}}$ be the Borel σ-algebra of $\mathbb{R}$. Suppose μ is a translation invariant measure on $\mathcal{B}_{\mathbb{R}}$ such that $\mu((0,1]) = 1$. Prove that $\mu = \lambda_{\mathcal{B}_{\mathbb{R}}}$, the restriction of the Lebesgue measure on $\mathcal{L}$ to $\mathcal{B}_{\mathbb{R}}$.

Proof. Notice first that we have $\mu((0,x]) = x$ for all $x \in \mathbb{Z}_{>0}$. To see this, we see that we can decompose $(0, x]$ into

$$\bigsqcup_{n=0}^{x-1} (n, n+1] = (0, x].$$

Hence

$$\mu((0,x]) = \sum_{n=0}^{x-1} \mu((n, n+1]).$$

Translation invariance tells us that $\mu((n, n+1]) = \mu((0,1]) = 1$, so we have

$$\mu((0,x]) = \sum_{n=0}^{x-1} 1 = x.$$

Translation invariance again tells us that

$$\mu((a,b]) = b - a$$

for a, b finite integers greater than 0. Similarly, we have

$$\mu((x, 0]) = x,$$

where $x \in \mathbb{Z}_{<0}$. Now, taking $n \in \mathbb{Z}_{>0}$, we wish to establish $\mu((0, 1/n]) = 1/n$. Notice that we can write

$$\bigsqcup_{j=0}^{n} \left(\frac{j}{n}, \frac{j+1}{n}\right].$$

Then we have

$$\mu\left(\bigsqcup_{j=0}^{n}\left(\frac{j}{n},\frac{j+1}{n}\right]\right)=\sum_{j=0}^{n}\mu\left(\left(\frac{j}{n},\frac{j+1}{n}\right]\right)=1.$$

Translation invariance tells us that all of these intervals have the same measure, though, and so we get

$$n\cdot\mu\left(\left(0,\frac{1}{n}\right]\right)=1\leftrightarrow\mu\left(\left(0,\frac{1}{n}\right]\right)=\frac{1}{n}.$$

We can further deduce using translation invariance that for all rational numbers a,b with $a<b$, we have $\mu((a,b])=b-a$. Now, take a,b real numbers with $a<b$. Then there is a rational number p such that $a<p<b$, so we have $(a,b]=(a,p]\sqcup(p,b]$. So

$$\mu((a,b])=\mu((a,p])+\mu((p,b]).$$

Now, recall that the rationals are dense in the reals. So we can form a sequence $\{q_n\}$ of rationals such that $q_n\geq a$ and $q_n\searrow a$. Hence, we have

$$(a,p]=\bigcup_{n=1}^{\infty}(q_n,p],$$

Continuity from below then gives us that

$$\mu((a,p])=\lim_{n\to\infty}\mu((q_n,p])=\lim_{n\to\infty}(p-q_n)=p-a.$$

Likewise, we can write

$$(p,b)=\bigcap_{n=1}^{\infty}(p,q_n],$$

where $\{q_n\}$ a sequence of rationals where $q_n\geq b$ and $q_n\searrow b$. Noticing $b\leq q_1<\infty$, we have $\mu((p,q_1])=q_1-p<\infty$, and so we can use continuity from above to deduce that

$$\mu((p,b])=\lim_{n\to\infty}\mu((p,q_n])=\lim_{n\to\infty}(q_n-p)=b-p.$$

Chaining these things together, then, we have

$$\mu((a,b])=(p-a)+(b-p)=b-a.$$

Let U be a bounded Borel set. Then we can cover it with a half open interval, letting $b=\sup(U)$ and $a=\inf(U)$, and noticing that $U\subset(a,b]$. Hence, $\mu(U)\leq\mu((a,b])=b-a<\infty$. So all bounded Borel sets are μ finite, and so by **Theorem 1.16** in the book, we get that for

$$F(x):=\begin{cases}\mu((0,x])\text{ if }x>0,\\0\text{ if }x=0,\\-\mu((x,0])\text{ if }x<0,\end{cases}$$

$\mu=\mu_F$ on $\mathcal{B}_{\mathbb{R}}$. But notice that

$$F(x):=\begin{cases}x\text{ if }x>0,\\0\text{ if }x=0,\\-x\text{ if }x<0,\end{cases}$$

or, in other words, $F(x)=x$. Hence, $\mu_F=\lambda_{\mathcal{B}_{\mathbb{R}}}$, and so we get that $\lambda_{\mathcal{B}_{\mathbb{R}}}=\mu$. □

Problem 18. Suppose $E\in\mathcal{L}$ is Lebesgue null, and $\varphi:\mathbb{R}\to\mathbb{R}$ is a C^1 function (continuous with continuous derivative). Prove that $\varphi(E)$ is also Lebesgue null.

Proof. We break this up into steps.
Step 1: Let I be a bounded open interval, say (a, b). Then we have that φ is Lipschitz on (a, b); to see this, take the closure, $[a, b]$. Since φ' is continuous, and this is compact, we have that it is bounded, which implies that φ is Lipschitz on $[a, b]$. Restricting down to (a, b), we get that φ is Lipschitz on (a, b), as desired.
Step 2: If I is a bounded open interval, we have $\lambda(\varphi(I)) \leq K\lambda(I)$, where $K \geq 0$ is a constant. Since I is a bounded interval, say (a, b), we have that $\lambda((a, b)) = b - a$. Since φ is continuous, we see it maps open intervals to intervals, and so we have $\lambda(\varphi(I)) = \varphi(b) - \varphi(a)$. By **Step 1**, since I is a bounded open interval, we have that φ is Lipschitz on it, and so

$$\lambda(\varphi(I)) = \varphi(b) - \varphi(a) \leq K(b - a),$$

where K is a real constant such that $K \geq 0$.
Step 3: Assume that E is a bounded set of Lebesgue measure 0. Since it is bounded, for fixed $\epsilon > 0$ we can find an open set G such that $\lambda(G) < \epsilon$. Furthermore, since E is bounded, we have $E \subset (a, b)$ for some interval. Since φ is Lipschitz on this interval, we get that we have a Lipschitz constant K associated to this interval, and the inequality descends down to the subintervals. Throughout, then, we use this associated K.

Notice as well we can cover G with disjoint open intervals, and so we get

$$\sum_{n=1}^{\infty} \lambda((a_n, b_n)) < \epsilon.$$

It suffices to show that, for all $\epsilon > 0$, we can cover $\varphi(E)$ with intervals where the sum of their measures is less than ϵ. By **Step 2**, using this global K on the interval covering E, we get that

$$\lambda(\varphi(E)) \leq \lambda\left(\bigsqcup_{n=1}^{\infty} \varphi((a_n, b_n))\right) = \sum_{n=1}^{\infty} \lambda(\varphi((a_n, b_n))) \leq K \sum_{n=1}^{\infty} \lambda((a_n, b_n)) < K\epsilon.$$

Since this applies for all $\epsilon > 0$, we get that

$$\lambda(\varphi(E)) = 0,$$

and so $\varphi(E)$ is Lebesgue null.
Step 4: Assume that E is unbounded and has Lebesgue measure 0. Recall that we can cover $\mathbb{R}$ with disjoint open intervals; that is,

$$\mathbb{R} = \bigsqcup_{n=1}^{\infty} (a_n, b_n).$$

Hence, we can write

$$E = E \cap \mathbb{R} = E \cap \bigsqcup_{n=1}^{\infty} (a_n, b_n) = \bigsqcup_{n=1}^{\infty} (E \cap (a_n, b_n)).$$

So, we have

$$\varphi(E) = \bigsqcup_{n=1}^{\infty} \varphi(E \cap (a_n, b_n)),$$

and furthermore

$$\lambda(\varphi(E)) = \sum_{n=1}^{\infty} \lambda(\varphi(E \cap (a_n, b_n))).$$

Notice that $E \cap (a_n, b_n)$ is bounded and is Lebesgue null by monotonicity. By **Step 3**, we have

$$\lambda(\varphi(E \cap (a_n, b_n))) = 0$$

for all n, and so

$$\lambda(\varphi(E)) = \sum_{n=1}^{\infty} 0 = 0.$$

Putting this all together, if E is a set which is Lebesgue null, and φ is a C^1 function, then $\varphi(E)$ is also Lebesgue null. □

Problem 19. Find an uncountable subset of $\mathbb{R}$ with Hausdorff dimension 0.

Proof.

Remark. I got inspiration for the solution from a Stackexchange thread
https://math.stackexchange.com/questions/1966537/hausdorff-dimension-of-a-cantor-set
though the details worked out are my own. It also seems to follow along a hint given in the recitation and discussion with other students.

Recall that we have the following definitions:

If (X, ρ) is a metric space, $p \geq 0$, $\epsilon > 0$, we define

$$\zeta_{p,\epsilon}(E) := \inf \left\{ \sum_{1}^{\infty} [\text{Diam}(B_n)]^p \;:\; \{B_n\} \text{ is a sequence of open balls, Diam}(B_n) \leq \epsilon \;\forall n,\; E \subset \bigcup_n B_n \right\},$$

where we have the convention $\inf \varnothing = \infty$. Furthermore, we define

$$\zeta_p(E) := \lim_{\epsilon \to 0} \zeta_{p,\epsilon}(E).$$

We showed in class that this is a metric outer measure. We define the Hausdorff dimension of a set $E \in \mathcal{B}_X$ to be

$$\text{HDim}(E) := \{\inf p \geq 0 \;:\; \zeta_p(E) = 0\}.$$

So to find a set with Hausdorff dimension 0, it suffices to show that for all $p > 0$, $\epsilon > 0$, the infimum of the sum of the diameter of open balls which cover E to the pth power, where the diameter less than or equal to ϵ, is 0. This suggests we think of a generalized Cantor set.

The standard Cantor set has Hausdorff dimension $\log(2)/\log(3)$, so it's clear this will not suffice. We again use the construction of the generalized Cantor set. Recall that, given some sequence $\{\alpha_n\}$ of numbers, we define $K_0 := [0, 1]$ and K_n to be the middle α_n segments removed from K_{n-1}, and our Cantor set is defined to be

$$K := \bigcap_n K_n.$$

Notice that step n of this sequence, we have 2^n intervals. In the standard Cantor construction, the length of each of these intervals is 3^{-n}; however, we are not restricted to taking powers of some number.

Taking away the middle $n/(n + 1)$ intervals, we see that we can get the Hausdorff dimension to be zero. Notice that the length at step n of the 2^n intervals is given by a recurrence relation;

$$l(n) = \frac{1}{2(n+1)\frac{1}{l(n-1)}},\; l(0) = 1.$$

Solving this recurrence relation gives the formula

$$l(n) = \frac{1}{(n+2)!2^n}.$$

Notice that for all $k > 0$,

$$\lim_{n\to\infty} 2^n l(n)^k = 0$$

Hence, for all $\epsilon > 0$, we can find N sufficiently large so that $l(N) < \epsilon$, and hence

$$\zeta_{p,\epsilon}(K) \leq 2^N l(N)^p.$$

Taking $\epsilon \to 0$ gives

$$\zeta_p(K) \leq \lim_{n\to\infty} 2^n l(n)^p = 0.$$

However, this applies for all $p > 0$, and so we get that

$$\inf\{p \geq 0 \ : \ \zeta_p(K) = 0\} = 0.$$

Hence, we see that

$$\text{HDim}(K) = 0,$$

or K is a set with Hausdorff dimension zero.

Since K is a generalized Cantor set, it shares the property that it is uncountable (since it is a perfect set), and so we get that K is an uncountable subset of $\mathbb{R}$ with Hausdorff dimension zero. $\square$

Problem 20. Suppose $(X, \mathcal{M})$ is a measurable space, and (Y, τ), (Z, θ) are topological spaces, $i : Y \to Z$ is a continuous injection which maps open sets to open sets (i.e. an open map), and $f : X \to Y$. Show that f is $\mathcal{M} - \mathcal{B}_\tau$ measurable if and only if $i \circ f$ is $\mathcal{M} - \mathcal{B}_\theta$ measurable.

Deduce that if $f : (X, \mathcal{M}) \to \overline{\mathbb{R}}$ only takes values in $\mathbb{R}$, then f is $\mathcal{M} - \mathcal{B}_{\overline{\mathbb{R}}}$ if and only if f is $\mathcal{M} - \mathcal{B}_{\mathbb{R}}$ measurable.

Proof. ($\Longrightarrow$) Assume f is $\mathcal{M} - \mathcal{B}_\tau$ measurable. Then this says that if $E \in \tau$, then $f^{-1}(E) \in \mathcal{M}$. To show that $i \circ f$ is $\mathcal{M} - \mathcal{B}_\theta$ measurable, we just need to show that, for all $E \in \theta$, $(i \circ f)^{-1} = f^{-1} \circ i^{-1}(E) \in \mathcal{M}$, since the open sets generate $\mathcal{B}_\theta$. Notice that since i is a continuous function, we get $i^{-1}(E)$ is open, i.e. $i^{-1}(E) \in \tau$. From our prior remark, $f^{-1}(i^{-1}(E)) \in \mathcal{M}$. Since this applies for all $E \in \theta$, we get the desired result.

($\Longleftarrow$) Suppose $i \circ f$ is $\mathcal{M} - \mathcal{B}_\theta$ measurable. Since i is an open map, for arbitrary $E \in \tau$ we have $i(E) = F \in \theta$. Since i is an injection, we have that it admits a left inverse, and so $i^{-1} \circ i(E) = E = i^{-1}(F)$. Hence, for all $E \in \tau$, there is an $F \in \theta$ such that $i^{-1}(F) = E$. So, taking $E \in \tau$ arbitrary, we get that the associated F pulls back to a measurable; i.e., $(i \circ f)^{-1}(F) \in \mathcal{M}$. But this implies that $f^{-1}(E) \in \mathcal{M}$ for all $E \in \tau$, and so we get that $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{B}_\tau$ by the proposition in the class notes, giving us the desired result.

We can take the natural inclusion $i : \mathbb{R} \to \overline{\mathbb{R}}$ via $i(x) = x$. This is clearly injective. We also see that this maps open balls to open balls, and so therefore maps open sets to open sets. Furthermore, $i^{-1}(\pm\infty) = \varnothing$, so we get that it pulls back open balls to open balls as well. Hence, i is a continuous injection which is also open, and since f only takes values to $\mathbb{R}$, we have the set up given in the problem. Hence, we deduce that f is $\mathcal{M} - \mathbb{B}_{\overline{\mathbb{R}}}$ measurable if and only if f is $\mathcal{M} - \mathcal{B}_{\mathbb{R}}$ measurable, where here we note that $i \circ f = f$. $\square$

Remark. Thomas O'Hare was a collaborator for this homework.

Problem 21. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(1) Show that a simple function
$$\psi = \sum_{k=1}^{n} c_k \chi_{E_k}$$
where $c_k > 0$ for all $k = 1, \ldots, n$ is integrable if and only if $\mu(E_k) < \infty$ for all $k = 1, \ldots, n$.

(2) Show that if a simple function
$$\psi = \sum_{k=1}^{n} c_k \chi_{E_k}$$
is integrable with $\mu(E_k) < \infty$ for all k, then
$$\int \psi = \sum_{k=1}^{n} c_k \mu(E_k).$$

Proof. (1) We have that ψ integrable and the linearity of integration for L^+ functions gives
$$\int \psi < \infty \iff \int \left(\sum_{k=1}^{n} c_k \chi_{E_k} \right) < \infty \iff \sum_{k=1}^{n} c_k \int \chi_{E_k} < \infty$$
$$\iff \sum_{k=1}^{n} c_k \mu(E_k) < \infty,$$
and since $c_k > 0$ this tells us that this is true if and only if $\mu(E_k) < \infty$ for $k = 1, \ldots, n$.

(2) Notice that we have
$$\int \psi = \int \sum_{k=1}^{n} c_k \chi_{E_k} = \sum_{k=1}^{n} c_k \int \chi_{E_k} = \sum_{k=1}^{n} c_k \mu(E_k)$$
by linearity of the integral for L^+ functions.

□

Problem 22. Suppose $f : (X, \mathcal{M}, \mu) \to [0, \infty]$ is $\mathcal{M}$-measurable and $\{f > 0\}$ is σ-finite. Show that there exists a sequence of simple function $\{\psi_n\}$ such that

- $\psi_n \nearrow f$,
- ψ_n is integrable for every $n \in \mathbb{N}$.

Proof. If $\{f > 0\}$ is σ-finite, we can cover it with disjoint measurable sets $\{X_n\}$ such that $\mu(X_n) < \infty$ for all n; that is, we have
$$\{f > 0\} = \bigsqcup_{n=1}^{\infty} X_n, \quad \mu(X_n) < \infty.$$

Define
$$E_n^k := \left\{ \frac{k-1}{2^n} < f \leq \frac{k}{2^n} \right\} \cap \left(\bigsqcup_{k=1}^{n} X_n \right),$$
$$F_n := \{2^n < f \leq \infty\} \cap \left(\bigsqcup_{k=1}^{n} X_n \right).$$

Then we set

$$\psi_n := \sum_{k=1}^{2^{2n}} \left(\frac{k-1}{2^n} \chi_{E_n^k} \right) + 2^n \chi_{F_n}.$$

We notice that ψ_n is integrable for every n, since $\mu(E_n^k) < \infty$ and $\mu(F_n) < \infty$. Furthermore, the proof of the lemma from class gives us that $\psi_n \nearrow f$. □

Problem 23. Assume Fatou's Lemma and prove the Monotone Convergence Theorem from it.

Proof. We want to prove if $\{f_n\} \subset L^+$ is an increasing sequence and $f = \lim_{n\to\infty} f_n = \sup_n f_n$, then

$$\int f = \lim_{n\to\infty} \int f_n.$$

Fatou's Lemma tells us that

$$\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n,$$

and we can note that $\lim_{n\to\infty} f_n = f = \liminf_{n\to\infty} f_n$, so we have

$$\int f \leq \liminf_{n\to\infty} \int f_n.$$

Moreover, we note that $f_n \leq f$ by assumption, so in particular we have

$$\int f_n \leq \int f$$

for all n by the monotonicity of integration. As a result, we have

$$\sup_{k\geq n} \int f_n \leq \int f.$$

Taking the limit as $n \to \infty$ of both sides, we have

$$\limsup_{n\to\infty} \int f_n \leq \int f,$$

and so

$$\limsup_{n\to\infty} \int f_n \leq \int f \leq \liminf_{n\to\infty} \int f_n.$$

In other words, we have

$$\lim_{n\to\infty} \int f_n = \int f,$$

as desired. □

Problem 24. Let $(X, \mathcal{M}, \mu)$ be a measure space.
(1) Suppose $f \in L^+$ and $\int f < \infty$. Prove that $\{f = \infty\}$ is μ-null and $\{f > 0\}$ is σ-finite.
(2) Suppose $f \in L^1(\mu, \mathbb{C})$. Prove that $\{f \neq 0\}$ is σ-finite.

Proof. (1) Let $A_n = \{f > n\}$. Then we have

$$\int_{A_n} n = n\mu(A_n) < \int_{A_n} f \leq \int f,$$

and so

$$\mu(A_n) < \frac{1}{n} \int f$$

for all n. Since $\int f = C < \infty$, we can write this as

$$\mu(A_n) < \frac{C}{n}$$

for all n. Since $A = \{f = \infty\} \subset \{f > n\} = A_n$, we have that $\mu(A) \leq \mu(A_n)$ for all n. So for all $\epsilon > 0$, $\mu(A) \leq \epsilon$, which implies $\mu(A) = 0$; that is, $\{f = \infty\}$ is μ-null.

Let $X_n = \{f \geq 1/n\}$. Then by an analogous argument, we have

$$\int_{X_n} \frac{1}{n} = \frac{\mu(X_n)}{n} \leq \int f,$$

so

$$\mu(X_n) \leq n \int f < \infty.$$

Moreover, notice that $\{f > 0\} = \bigcup_{j=1}^{\infty} X_j$, where $\mu(X_j) < \infty$. So $\{f > 0\}$ is σ-finite.

(2) We repeat the argument above, except for $X_n = \{|f| \geq 1/n\}$. Hence, we have

$$\int_{X_n} \frac{1}{n} = \frac{\mu(X_n)}{n} \leq \int |f| < \infty,$$

and so we get again

$$\mu(X_n) \leq n \int |f| < \infty,$$

and so

$$\{|f| > 0\} = \{f \neq 0\} = \bigcup_{n=1}^{\infty} X_n, \quad \mu(X_n) < \infty.$$

Hence, $\{f \neq 0\}$ is σ-finite.

□

Problem 25. Suppose $(X, \mathcal{M}, \mu)$ is a measure space and $f \in L^1(\mu, \mathbb{C})$. Prove that for every $\epsilon > 0$, there exists a $\delta > 0$ such that for every $E \in \mathcal{M}$ with $\mu(E) < \delta$,

$$\int_E |f| < \epsilon.$$

Proof.

Remark. This was adapted from a proof in Royden and Fitzpatrick.

We have $f \in L^1(\mathbb{C}, \mu)$, so $|f| \in L^+$. Fix $\epsilon > 0$, then we have there exists a $\psi \in \mathrm{SF}^+$ such that

$$0 \leq \int |f| - \int \psi < \frac{\epsilon}{2}.$$

Now notice that $\psi \in \mathrm{SF}^+$ tells us that there exists an M such that $0 \leq \psi \leq M$ on X. Hence, we have

$$\int_E |f| = \int_E \psi + \int_E (|f| - \psi) < M\mu(E) + \frac{\epsilon}{2}.$$

Solving

$$M\delta + \frac{\epsilon}{2} = \epsilon,$$

we get

$$\delta = \frac{\epsilon}{2M}.$$

So, if $E \in \mathcal{M}$ is such that $\mu(E) < \delta$, then we have that

$$\int_E |f| < \epsilon,$$

as desired. □

Remark. Thomas O'Hare was a collaborator.

Problem 26. Let μ be a Lebesgue-Stieltjes Borel measure on $\mathbb{R}$. Show that $C_c(\mathbb{R})$, the continuous functions of compact support, is dense in $\mathcal{L}^1(\mu, \mathbb{R})$. Does the same hold for $\overline{\mathbb{R}}$ and $\mathbb{C}$-valued functions?

Proof. We follow the proof of **Theorem 2.26**
Step 1: We want to show that the integrable simple functions are dense in $\mathcal{L}^1(\mu, \mathbb{R})$. Take $f \in \mathcal{L}^1(\mu, \mathbb{R})$. Then from prior problems/the lecture notes (**Theorem 2.10 b**), we can construct a sequence of integrable simple functions such that $|\phi_n| \nearrow |f|$, since f integrable implies that $\{f \neq 0\}$ is σ-finite. Furthermore, we have $|\phi_n - f| \leq |\phi_n| + |f| \leq 2|f|$, and since $f \in \mathcal{L}^1(\mu, \mathbb{R})$ we have $|f| \in \mathcal{L}^1(\mu, \mathbb{R})$. The dominated convergence theorem then tells us that

$$\lim \int |\phi_n - f| d\mu = \int \lim |\phi_n - f| d\mu = 0.$$

Thus, we have that these integrable simple functions are dense in $\mathcal{L}^1(\mu, \mathbb{R})$.
Step 2: We now want to show that we can use continuous functions with compact support to get arbitrarily close to simple integrable functions. Write $\phi_n = \sum_{j=1}^N a_j \chi_{E_j}$. We have that $\mu(E_j) = |a_j|^{-1} \int_{E_j} |\phi| \leq |a_j|^{-1} \int |f| < \infty$ for all j. Examine χ_E for E measurable where $\mu(E) < \infty$. We use **Proposition 1.20** from the book, which tells us that if E measurable with $\mu(E) < \infty$, then for every $\epsilon > 0$ we can find a set A that is a finite union of open intervals such that

$$\int |\chi_E - \chi_A| = \mu(E \Delta A) < \epsilon.$$

To see this, notice that **Theorem 1.18** tells us that $\mu(E)$ is the infimum over open sets containing it and the supremum over compact sets contained in it. Recall that

$$E \Delta A = (E - A) \sqcup (A - E).$$

Take $K \subset E \subset U$ such that K compact, U open, and

$$\mu(U - E) < \epsilon/2, \quad \mu(E - K) < \epsilon/2.$$

Since U is open, we can write it as a union of disjoint open intervals, $U = \bigsqcup_{i=1}^\infty I_i$. Hence, we have $K \subset \bigsqcup_{i=1}^\infty I_i$, and so using the compactness of K we have $K \subset \bigsqcup_{i=1}^N I_i$. Write $A = \bigsqcup_{i=1}^n I_i$. Then we have $K \subset A \subset U$, and furthermore we have

$$E - A \subset E - K,$$
$$A - E \subset U - E,$$

and so

$$\mu(E \Delta A) = \mu(E - A) + \mu(A - E) \leq \mu(E - K) + \mu(U - E) < \epsilon/2 + \epsilon/2 = \epsilon.$$

So for all ϵ, we can find this desired construction. Since we have that

$$\mu(E \Delta F) = \int |\chi_E - \chi_F|,$$

we can approximate χ_E in $\mathcal{L}^1$ for $\mu(E) < \infty$ with χ_{A_n}, where A_n is a finite disjoint union of open intervals.
Step 3: We can approximate open intervals $I_k = (a, b)$ with continuous functions (h_ϵ) in $\mathcal{L}^1$, where

$h_\epsilon = 1$ on $[a+\epsilon/2, b-\epsilon/2]$, $h_j = 0$ on $(\infty, a]$ and $[b, \infty)$, and is a linear interpolation between 0 and 1 on $[a, a+\epsilon/2]$ and $[b-\epsilon/2, b]$. Notice that in $\mathcal{L}^1$, we have that

$$\int |h_\epsilon - \chi_{I_k}| < \epsilon,$$

and these h_ϵ have compact support. So we can approximate χ_{I_k} by a continuous function with compact support in $\mathcal{L}^1$.

Step 4: Let

$$\psi = \sum_{i=1}^{n} a_n \chi_{E_n}$$

be an integrable simple function given as in **Step 1**. Let $M = \max\{|a_i|\}_{i=1}^n$; this is finite by the observation made in **Step 1**. For each i we can find A_i a finite disjoint union of open intervals such that

$$\int |\chi_{E_i} - \chi_{A_i}| < \frac{\epsilon}{2Mn}$$

by **Step 2**. Write $A_i = \bigsqcup_{j=1}^{k_i} I_{i,j}$. Let $J = \max\{k_i\}_{i=1}^n$. By **Step 3**, for each (i,j), we can find a continuous function $h_{i,j}$ such that

$$\int |h_{i,j} - \chi_{I_{i,j}}| < \frac{\epsilon}{2MJn}.$$

Let $h = \sum_{i=1}^n a_i \left(\sum_{j=1}^{k_i} h_{i,j}\right) = \sum_{i=1}^n a_i h_i$, where $h_i = \sum_{j=1}^{k_i} h_{i,j}$. Notice that h is continuous, since it is a linear combination of continuous functions. Furthermore, we see that this choice of h gives us

$$\int |h-\psi| = \int \left|\sum_{i=1}^n a_i(\chi_{E_i} - h_i)\right| \le \sum_{i=1}^n |a_i| \int |\chi_{E_i} - h_i| \le \sum_{i=1}^n |a_i| \int |\chi_{E_i} - \chi_{A_i}| + \sum_{i=1}^n |a_i| \int \left|\sum_{j=1}^{k_i} \chi_{I_{i,j}} - h_{i,j}\right|$$

$$\le \sum_{i=1}^n |a_i| \int |\chi_{E_i} - \chi_{A_i}| + \sum_{i=1}^n |a_i| \sum_{j=1}^{k_i} \int |\chi_{I_{i,j}} - h_{i,j}| < \epsilon.$$

We also see that h has compact support. Notice that if $f, g \in C_c(\mathbb{R})$, then $f+g \in C_c(\mathbb{R})$, since $\mathrm{Supp}(f+g) \subset \mathrm{Supp}(f) \cup \mathrm{Supp}(g)$, and so $\mathrm{Supp}(f+g)$ is bounded and closed and hence compact, and for $\alpha \neq 0$, we have $\mathrm{Supp}(\alpha f) = \mathrm{Supp}(f)$, so we get h has compact support.

Thus, we have that for all integrable simple functions ψ, we can construct a sequence (h_n) of continuous functions with compact support such that $h_n \to \psi$ in $\mathcal{L}^1$.

Step 5: Finally, we wish to use all of these approximations to show that we an approximate our integrable function f in $\mathcal{L}^1$. That is, for all $\epsilon > 0$, we can find a function with compact support h such that

$$\int |h - f| \to 0.$$

Fix $\epsilon > 0$. From **Step 1**, we have that we can find an integrable simple function such that

$$\int |\psi - f| < \frac{\epsilon}{2}.$$

From **Step 4**, we can find a continuous function h with compact support such that

$$\int |h - \psi| < \frac{\epsilon}{2}.$$

Hence, we have that for this choice of $\epsilon > 0$, we get that we can find a continuous function h such that

$$\begin{aligned}\int |h - f| &= \int |h - \psi + \psi - f| \\ &\le \int |h - \psi| + \int |\psi - f| < \epsilon.\end{aligned}$$

Since this applies for all $\epsilon > 0$, we can construct a sequence (h_n) such that $h_n \to f$ in $\mathcal{L}^1$, and furthermore we see that our choices of h_n are such that they have compact support. Since the choice of f was arbitrary, we get that $C_c(\mathbb{R})$ is dense in $\mathcal{L}^1(\mu, \mathbb{R})$.

For $\mathbb{C}$, the same argument applies; we can approximate the real and imaginary parts with simple functions by the same theorem, approximate those simple functions with continuous functions with compact support, and therefore approximate the integrable functions with continuous functions with compact support.

For $\overline{\mathbb{R}}$, we have it's finite almost everywhere, so define $\widetilde{f} = f$ where f is finite and $\widetilde{f} = 0$ where f is infinite. Then the integral is the same, and we can apply the previous cases to $\widetilde{f}$ to get compact functions which approximate it in $\mathcal{L}^1$, and therefore also approximate f in $\mathcal{L}^1$. □

Problem 27. Suppose $f : [a, b] \to \mathbb{C}$ is Lebesgue measurable and $\epsilon > 0$. There is a compact set $E \subset [a, b]$ such that $\lambda(E^c) < \epsilon$ and $f|_E$ is continuous.

Proof.

Remark. Adapted from a proof from old notes.

Step 1: Let

$$f = \sum_{i=1}^{N} a_i \chi_{E_i}$$

be a simple measurable function on $[a, b] \to \mathbb{C}$, where $[a, b] = \bigsqcup_{i=1}^{N} E_i$. For each i, choose a closed subset $F_i \subset E_i$ such that

$$\mu(E_i - F_i) < \frac{\epsilon}{N},$$

which we can do by inner regularity. Let $F = \bigsqcup_{i=1}^{N} F_i$; F is closed since we took a finite union of closed sets. We get

$$\mu(E - F) = \mu\left(\bigsqcup_{i=1}^{n} (E_i - F_i)\right) = \sum_{i=1}^{N} \mu(E_i - F_i) < \sum_{i=1}^{N} \frac{\epsilon}{N} = \epsilon.$$

The F_i are closed and disjoint, so we get

$$\lim_{\substack{x \to x_0 \\ x \in F}} f(x) = f(x_0),$$

since $x_0 \in F_i$ for some i, and so for any sequence $x_n \to x_0$, we must have $x_n \in F_i$ for n large enough, and so we must have

$$\lim_{\substack{x \to x_0 \\ x \in F}} f(x) = a_i = f(x_0).$$

Since this holds for all $x_0 \in F$, we have f is continuous relative to F.

Step 2: Take f measurable. We can construct ψ_n such that $\psi_n \to f$ pointwise and the ψ_n are simple measurable functions. Since each ψ_n satisfies the property, we can pick $F_n \subset [a, b]$ closed such that

$$\mu([a, b] - F_n) < \frac{\epsilon}{2^{n+1}}.$$

Take as well $F_0 \subset [a,b]$ such that
$$\mu([a,b]-F_0) < \frac{\epsilon}{2}$$
and $\psi_n \to f$ uniformly on F_0, which we can do by Egoroff's theorem, and take it to be closed, which we can do using the inner regularity of the measure. Take $E = \bigcap_{k=0}^{\infty} F_k \subset [a,b]$. Then E is closed, and $\mu([a,b]-E) < \epsilon$. Since $\psi_n \to f$ uniformly on E, ψ_n continuous on E for all n by **Step 1**, we have that f is continuous on E. Since E is closed and bounded, it is compact. □

Problem 28. Suppose $f \in \mathcal{L}^1([0,1],\lambda)$ is an integrable non-negative function.
(1) Show that for every $n \in \mathbb{N}$, $\sqrt[n]{f} \in \mathcal{L}^1([0,1],\lambda)$.
(2) Show that $(\sqrt[n]{f})$ converges in $\mathcal{L}^1$ and compute its limit.

Proof. (1) We can write
$$\int_{[0,1]} \sqrt[n]{f} d\lambda = \int_{\{f\geq 1\}} \sqrt[n]{f} d\lambda + \int_{\{f<1\}} \sqrt[n]{f} d\lambda$$
and note that this is bounded above by
$$\int_{[0,1]} \sqrt[n]{f} d\lambda \leq \int_{\{f\geq 1\}} \sqrt[n]{f} d\lambda + 1.$$
Notice that $\sqrt[n]{x} \leq x$ for $x \geq 1$, so we have
$$\int_{\{f\geq 1\}} \sqrt[n]{f} d\lambda \leq \int_{\{f\geq 1\}} f d\lambda \leq \int f d\lambda < \infty.$$
Hence, we have
$$\int_{[0,1]} \sqrt[n]{f} d\lambda < \infty,$$
or $\sqrt[n]{f}$ is integrable.

(2) We'd like to show that $\sqrt[n]{f} \to \chi_{f\neq 0}$ in $\mathcal{L}^1$. Write this as
$$\lim_{n\to\infty} \int_{[0,1]} |\sqrt[n]{f} - \chi_{f\neq 0}| d\lambda = \lim_{n\to\infty} \int_{\{f\geq 1\}} |\sqrt[n]{f} - \chi_{f\neq 0}| d\lambda + \lim_{n\to\infty} \int_{\{f<1\}} |\sqrt[n]{f} - \chi_{f\neq 0}| d\lambda.$$
Notice that based on the bounds we may write this as
$$\lim_{n\to\infty} \left(\int_{\{f\geq 1\}} (\sqrt[n]{f} - 1) d\lambda + \int_{\{0<f<1\}} (1 - \sqrt[n]{f}) d\lambda \right),$$
where we drop the case where $f = 0$, since both are equal to 0 and give us 0 integral. For the left integral, we have
$$|\sqrt[n]{f} - 1| \leq \sqrt[n]{f} \leq f$$
on the domain $\{f \geq 1\}$, and so we can use the dominated convergence theorem to write this as
$$\lim_{n\to\infty} \int_{\{f\geq 1\}} (\sqrt[n]{f} - 1) d\lambda = 0.$$
On the right, we have
$$|1 - \sqrt[n]{f}| \leq 1,$$
on the domain $\{0 < f < 1\} \subset [0,1]$, which is integrable, and so therefore the dominated convergence theorem again gives us
$$\lim_{n\to\infty} \int_{\{f<1\}} (1 - \sqrt[n]{f}) d\lambda = 0.$$

Hence, we have

$$\lim_{n\to\infty}\int_{[0,1]} |\sqrt[n]{f} - \chi_{f\neq 0}| d\lambda = 0,$$

and so it converges in $\mathcal{L}^1$ to 1.

□

Problem 29. Suppose $(X, \mathcal{M}, \mu)$ is a measure space and $f_n \to f$ in measure and $g_n \to g$ in measure. Show that

(1) $|f_n| \to |f|$ in measure.
(2) $f_n + g_n \to f + g$ in measure.
(3) $f_n g_n \to fg$ if $\mu(X) < \infty$ but not necessarily if $\mu(X) = \infty$.

Proof. (1) We want to show that for all $\epsilon > 0$,

$$\mu(\{||f_n| - |f|| \geq \epsilon\}) \to 0.$$

Notice that the reverse triangle inequality gives

$$||f_n| - |f|| \leq |f_n - f|,$$

and so we get

$$\{||f_n| - |f|| \geq \epsilon\} \subset \{|f_n - f| \geq \epsilon\},$$

and so

$$\mu(\{||f_n| - |f|| \geq \epsilon\}) \leq \mu(\{|f_n - f| \geq \epsilon\}) \to 0.$$

Thus, we have convergence in measure.

(2) The triangle inequality gives us, for all $\epsilon > 0$,

$$\{|f_n + g_n - f - g| \geq \epsilon\} \subset \{|f_n - f| + |g_n - g| \geq \epsilon\} = \{|f_n - f| \geq \epsilon/2\} \cup \{|g_n - g| \geq \epsilon/2\}.$$

Since $f_n \to f$ and $g_n \to g$ in measure, we have that the measure of both of these on the right go to 0, and so

$$\mu(\{|f_n + g_n - f - g| \geq \epsilon) \leq \mu(\{|f_n - f| \geq \epsilon/2\}) + \mu(\{|g_n - g| \geq \epsilon/2\}) \to 0.$$

Since the choice of $\epsilon > 0$ was arbitrary, we have $f_n + g_n \to f + g$ in measure.

(3)

Remark. Solution adapted from the old class notes.

Since $\mu(X) < \infty$, we can write

$$X = \bigcup_{n=1}^{\infty} \{|f| \leq n\},$$

The continuity of measure gives

$$\infty > \mu(X) = \lim_{n\to\infty} \mu(\{|f| \leq n\}),$$

and so therefore we have that the tails must go to 0; that is,

$$\lim_{n\to\infty} \mu(\{|f| > n\}) = 0.$$

Hence, for all $\zeta > 0$, we can choose M_1 sufficiently large so that we get

$$\mu(\{|f| > M_1\}) < \zeta.$$

We can do an analogous argument for $|g|$. Taking $M = \max(M_1, M_2)$, we have that these functions are bounded by M outside sets of arbitrarily small measure.

We can write

$$f_n g_n - fg = (f_n - f)(g_n - g) + f(g_n - g) + g(f_n - f).$$

This gives us that

$$\{|f_ng_n - fg| \geq \epsilon\} \subset \{|f_n - f||g_n - g| \geq \epsilon/3\} \cup \{|f||g_n - g| \geq \epsilon/3\} \cup \{|g||f_n - f| \geq \epsilon/3\}.$$

Notice as well that

$$\{|f_n - f||g_n - g| \geq \epsilon/3\} \subset \{|f_n - f| \geq \sqrt{\epsilon/3}\} \cup \{|g_n - g| \geq \sqrt{\epsilon/3}\},$$

and since these converge in measure we have that this goes to 0 as $n \to \infty$. So without loss of generality, we can take this term to be 0. For the remaining terms, we can use the fact we derived earlier. Choose $\zeta > 0$. Let F_1 be the set where f is bounded by M and F_2 be the set where g is bounded by M, where M is chosen such that $\mu(F_i^C) < \zeta/2$ for $i \in \{1, 2\}$. Then we have

$$\{|f||g_n - g| \geq \epsilon/3\} \subset \{|g_n - g| \geq \epsilon/3M\} \cup F_1^C,$$

$$\{|g||f_n - f| \geq \epsilon/3\} \subset \{|f_n - f| \geq \epsilon/3M\} \cup F_2^C,$$

and so chaining all these together and using the fact that $f_n \to f$ and $g_n \to g$ in measure, we get

$$\lim_{n\to\infty} \mu(\{|f_ng_n - fg| \geq \epsilon\}) \leq \zeta.$$

Since the choice of $\zeta > 0$ was arbitrary, we can let $\zeta \to 0$. Since ϵ was arbitrary, we get that it works for all $\epsilon > 0$. Hence, it converges in measure.

If we assume $\mu(X) = \infty$, we do not necessarily have that these functions are bounded. Take $f_n(x) = x^2 + (1/n)\chi_{(n,n+1)}$, $g_n(x) = (1/n)\chi_{(n,n+1)}$. We have $g_n(x) \to 0$, $f_n(x) \to x^2$ in measure, and for $x \in (n, n + 1)$, $n > 1$,

$$|f_ng_n - fg| = x^2(1/n) + (1/n^2) > n + 1/n^2 > 1.$$

So we do not have convergence in measure, since $\mu(\{|f_ng_n - fg| \geq 1\} = 1$ for all $n > 1$.

□

Problem 30. Suppose $(X, \mathcal{M}, \mu)$ is a measure space and $f_n \to f$ in measure.

(1) Show that if $f_n \geq 0$ everywhere, then $\int f \leq \liminf \int f_n$.
(2) Suppose $|f_n| \leq g \in \mathcal{L}^1$. Prove that $\int f = \lim \int f_n$ and $f_n \to f$ in $\mathcal{L}^1$.

Proof. (1) Fatou's Lemma gives

$$\int \liminf f_n \leq \liminf \int f_n,$$

since $f_n \geq 0$. Thus, we can construct a subsequence $f_{n_j} \to \liminf f_n$, and so we get

$$\int \lim_j f_{n_j} = \int \liminf f_n \leq \liminf \int f_n.$$

Now, since $f_n \to f$ in measure, we have $f_{n_j} \to f$ in measure as well, so we can construct a subsequence $f_{n_{j_k}} \to f$ almost everywhere. Hence, we have

$$\int f = \int \lim_k f_{n_{j_k}} = \int \lim_j f_{n_j} \leq \liminf \int f_n.$$

(2) It suffices to do this for real valued functions, since if $f_n \to f$ in measure, we have

$$|f_n - f| \leq |\mathrm{Re}(f_n) - \mathrm{Re}(f)| + |\mathrm{Im}(f_n) - \mathrm{Im}(f)| \leq 2|f_n - f|,$$

and so $f_n \to f$ in measure if and only if $\mathrm{Re}(f_n) \to \mathrm{Re}(f)$ and $\mathrm{Im}(f_n) \to \mathrm{Im}(f)$ converge in measure, and so we can consider both separately.

If $|f_n| \le g \in \mathcal{L}^1$, we have $f_n \le g$ and $-f_n \le g$, or in other words, $0 \le g - f_n$ and $0 \le g + f_n$. Using (1), we get

$$\int g - \int f = \int (g - f) \le \liminf \int (g - f_n) = \int g - \limsup \int f_n,$$

and

$$\int g + \int f = \int (g + f) \le \liminf \int (g + f_n) = \int g + \liminf \int f_n.$$

Since $g \in \mathcal{L}^1$, we can subtract it from both sides and rearrange terms to get

$$\limsup \int f_n \le \int f \le \liminf \int f_n,$$

or that

$$\lim \int f_n = \int f.$$

To see that $f_n \to f$ in $\mathcal{L}^1$, we need to show that $\int |f_n - f| \to 0$. Notice that $f_n \to f$ in measure implies $|f_n - f| \to 0$ in measure as well, and so we can use this and $h = g + |f| \ge |f_n| + |f| \ge |f_n - f|$ to get that, by what we've just shown,

$$\lim \int |f_n - f| = \int 0 = 0.$$

Hence, $f_n \to f$ in $\mathcal{L}^1$.

$\square$

Remark. Thomas O'Hare was a collaborator.

Problem 31. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is such that each x-section f_x is Borel measurable and f^y is continuous. Show that f is Borel measurable.

Proof.

Remark. Followed the solution given in
https://math.stackexchange.com/questions/647235/counterexample-to-measurable-in-each-variable-separately-implies-measurable
which also matches closely the hint given by Stefan in recitation.

We define a sequence of functions $f_n(x, y) : \mathbb{R}^2 \to \mathbb{R}$ for $x \in [i/n, (i+1)/n)$, $i \in \mathbb{Z}$ by

$$f_n(x, y) = f\left(\frac{i}{n}, y\right).$$

Fix $\epsilon > 0$. Since we have f^y is continuous, we have that at each $x = i/n$ it's continuous, and so we can find a δ such that

$$\left|x - \frac{i}{n}\right| < \delta \implies \left|f(x, y) - f\left(\frac{i}{n}, y\right)\right| < \epsilon.$$

So, for any $n > 1/\delta$, any fixed $(x, y) \in \mathbb{R}^2$ where $x \in [i/n, (i+1)/n)$, we get

$$\left|x - \frac{i}{n}\right| < \frac{1}{n} < \delta,$$

so

$$|f_n(x, y) - f(x, y)| = \left|f\left(\frac{i}{n}, y\right) - f(x, y)\right| < \epsilon.$$

Since ϵ fixed was arbitrary, we get that $f_n \to f$ pointwise. We then need to check that f_n is measurable; that is, $\{f_n > a\}$ is measurable for each $a \in \mathbb{R}$. But notice that

$$\{f_n > a\} = \bigcup_{i \in \mathbb{Z}} \left(\left[\frac{i}{n}, \frac{i+1}{n}\right) \times \left\{y \in \mathbb{R} \ : \ f\left(\frac{i}{n}, y\right) > a\right\}\right),$$

which is a union of measurable sets, since x-sections are measurable, and so $\{f_n > a\}$ is measurable. Hence, we have that f is a limit of measurable functions, and so it's measurable. □

Problem 32. Suppose $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are measurable spaces and $(E_n) \subset \mathcal{M} \times \mathcal{N}$. Prove the following assertions about x-sections.

(1)
$$\left[\bigcup E_n\right]_x = \bigcup (E_n)_x,$$

(2)
$$\left[\bigcap E_n\right]_x = \bigcap (E_n)_x,$$

(3) $(E_m - E_n)_x = (E_m)_x - (E_n)_x$,

(4) $\chi_{E_n}(x, y) = \chi_{(E_n)_x}(y)$ for all $x \in X$ and $y \in Y$.

Proof. (1) Recall that we define

$$\left[\bigcup E_n\right]_x = \left\{y \in Y \ : \ (x, y) \in \bigcup E_n\right\}.$$

Take $y \in \{y \in Y \ : \ (x,y) \in \bigcup E_n\}$. Then we have $(x,y) \in \bigcup E_n$, or in other words $(x,y) \in E_n$ for some n. But this implies that $y \in \bigcup\{y \in Y \ : \ (x,y) \in E_n\} = \bigcup(E_n)_x$. Hence we have

$$\left[\bigcup E_n\right]_x \subset \bigcup(E_n)_x.$$

Now take $y \in \bigcup(E_n)_x$. By definition, we have $(x,y) \in E_n$ for some n, but this says $(x,y) \in \bigcup E_n$, or $y \in \{y \in Y \ : \ (x,y) \in \bigcup E_n\}$. Hence, we have

$$\bigcup(E_n)_x \subset \left[\bigcup E_n\right]_x,$$

and so we have equality.

(2) Let $y \in [\bigcap E_n]_x$. Then we have $(x,y) \in \bigcap E_n$, which says that $(x,y) \in E_n$ for all n, or $y \in \bigcap\{y \in Y \ : \ (x,y) \in E_n\}$. Hence,

$$\left[\bigcap E_n\right]_x \subset \bigcap(E_n)_x.$$

Now, take $y \in \bigcap(E_n)_x$. Then again, for all n we have $(x,y) \in E_n$, which tells us that $(x,y) \in \bigcap E_n$. So we get

$$\bigcap(E_n)_x \subset \left[\bigcap E_n\right]_x.$$

Thus, we have equality.

(3) Take $y \in (E_m - E_n)_x = \{y \ : \ (x,y) \in E_m - E_n\}$. Then we have that $(x,y) \in E_m$, $(x,y) \notin E_n$, so we have that $y \in (E_m)_x \cap ((E_n)_x)^C = (E_m)_x - (E_n)_x$, or $(E_m - E_n)_x \subset (E_m)_x - (E_n)_x$. Taking $y \in (E_m)_x - (E_n)_x = \{y \ : \ (x,y) \in E_m\} \cap \{y \ : \ (x,y) \notin E_m\}$, we have y is such that $(x,y) \in E_m$ and $(x,y) \notin E_n$, which tells us that $(x,y) \in E_m - E_n$, or $y \in (E_m - E_n)_x$. Hence, we have equality.

(4) Assume $\chi_{E_n}(x,y) = 1$. Then we have that $(x,y) \in E_n$, and so we get $y \in (E_n)_x$, which implies that $\chi_{(E_n)_x}(y) = 1$. Assume $\chi_{E_n}(x,y) = 0$. Then we have $(x,y) \notin E_n$, or $y \notin (E_n)_x$, and so $\chi_{(E_n)_x}(y) = 0$. Hence, $\chi_{E_n}(x,y) = \chi_{(E_n)_x}(y)$.

□

Problem 33.

(1) Let $X = Y = [0,1]$, $\mathcal{M} = \mathcal{N} = \mathcal{B}_{[0,1]}$, $\mu = \lambda$ Lebesgue measure, and ν counting measure. Let $\Delta = \{(x,x) \ : \ x \in [0,1]\}$ be the diagonal. Prove that

$$\int\int \chi_\Delta d\mu d\nu,$$

$$\int\int \chi_\Delta d\nu d\mu,$$

and

$$\int \chi_\Delta d(\mu \times \nu)$$

are all unequal.

(2) Let $X = Y = \mathbb{N}$, $\mathcal{M} = \mathcal{N} = \mathcal{P}(\mathbb{N})$, and $\mu = \nu$ counting measure. Define

$$f(m,n) = \begin{cases} 1 \text{ if } m = n, \\ -1 \text{ if } m = n+1 \\ 0 \text{ otherwise.} \end{cases}$$

Prove that

$$\int |f| d(\mu \times \nu) = \infty$$

and

$$\int\int f d\mu d\nu$$

and

$$\int\int f d\nu d\mu$$

both exist and are unequal.

Proof.

(1) We have $\Delta \subset [0,1]^2$ is closed, since it is closed under sequential limits, and so $\Delta \in \mathcal{B}_{[0,1]^2}$. By a proposition from class, we have that this tells us that $\Delta \in \mathcal{B}_{[0,1]} \times \mathcal{B}_{[0,1]}$, and so we can calculate

$$\int \chi_\Delta d(\mu \times \nu) = (\mu \times \nu)(\Delta).$$

Notice that the outer measure construction gives us

$$(\mu \times \nu)(\Delta) = \inf\left\{\sum_{n=1}^{\infty} \mu(A_n)\nu(B_n) \ : \ \Delta \subset \bigcup_{n=1}^{\infty}(A_n \times B_n), A_n \in \mathcal{M}, B_n \in \mathcal{N}.\right\}$$

Observe as well that for any such cover, we have

$$\Delta \subset \bigcup_{n=1}^{\infty}(A_n \times B_n).$$

Since $\Delta = \{(x,x) \ : \ x \in [0,1]\}$, we get that this is the same as

$$[0,1] \subset \bigcup_{n=1}^{\infty}(A_n \cap B_n).$$

Since $A_n, B_n \subset [0,1]$, taking the Lebesgue measure of both sides gives

$$\lambda([0,1]) = 1 \leq \sum_{n=1}^{\infty} \lambda(A_n \cap B_n).$$

This implies that, for some N, we have that

$$\lambda(A_N \cap B_N) > 0 \implies \lambda(A_N) > 0, \lambda(B_N) > 0.$$

Recall that $\lambda(B_N) > 0$ implies that $\nu(B_N) = \infty$, since we have shown that B_N finite/countable implies $\lambda(B_N) = 0$. So, this tells us that for all possible covers of Δ, we must have that there is an N such that

$$(\mu \times \nu)(A_N \times B_N) = \infty.$$

Hence, we have

$$\int \chi_\Delta d(\mu \times \nu) = (\mu \times \nu)(\Delta) = \infty.$$

Now fix some $x \in X$. Then we have

$$\int_Y \chi_\Delta(x,y) d\nu(y) = \nu(\{x\}) = 1,$$

and so

$$\int_X \int_Y \chi_\Delta(x,y) d\nu(y) d\mu(x) = \int_X d\mu(x) = \mu([0,1]) = 1.$$

Finally, fix some $y \in Y$. Then we have

$$\int_X \chi_\Delta(x,y) d\mu(x) = \mu(\{y\}) = 0,$$

and so
$$\int_Y \int_X \chi_\Delta(x,y)d\mu(x)d\nu(y) = \int_Y 0 d\nu(y) = 0.$$
So we have they are all unequal.

Remark. Throughout the next two parts, I use the fact that integrating with counting measures gives sums. This can be seen by setting ν to be the counting measure and noticing
$$\int_{\mathbb{N}} f(x)d\nu(x) = \sum_{n=1}^{\infty} \int_{\{n\}} f(x)d\nu(x) = \sum_{n=1}^{\infty} f(n),$$
since $\mathbb{N} = \bigsqcup_{n=1}^{\infty}\{n\}$.

(2) We first show that
$$\int |f| d(\mu \times \nu) = \infty.$$
Notice that for sets $U, V \subset \mathbb{N}$, we have $(\mu\times\nu)(U\times V) = \mu(U)\nu(V)$, since it's just measure the sizes of each set. We then write
$$E = \bigcup_{n=1}^{\infty} \{(n,n)\} \cup \{(n, n+1)\},$$
and this is such that $|f| = \chi_E$. Hence, we see
$$\int |f| d(\mu\times\nu) = (\mu\times\nu)(E) = \sum_{n=1}^{\infty} 2 = \infty.$$
Next, we want to calculate $\int\int f d\mu d\nu$ and $\int\int f d\nu d\mu$. For the first, we write it as
$$\int_Y \int_X f(x,y)d\mu(x)d\nu(y).$$
Fix $y \in Y = \{1, 2, \ldots\}$, then we have
$$\int_X f(x,y)d\mu(x) = f(y,y) + f(y+1, y) = 0.$$
Hence, we get
$$\int_Y \int_X f(x,y)d\mu(x)d\nu(y) = \sum_{y=1}^{\infty} 0 = 0.$$
Now, fix $x \in X = \{1, 2, \ldots\}$. Then we have
$$\int_Y f(x,y)d\nu(y) = f(x,x) + f(x, x-1) = \begin{cases} 1 \text{ if } x = 1 \\ 0 \text{ otherwise,} \end{cases}$$
since if $x = 1$ there is no $y = x - 1 = 0 \in \{1, 2, \ldots\}$, and so we just have $f(1,1) = 1$. So we can write this as
$$\int_X \int_Y f(x,y)d\nu(y)d\mu(x) = \sum_{x=1}^{\infty}\sum_{y=1}^{\infty} f(x,y) = 1.$$
We have that these are not equal.

□

Problem 34. Show that the conclusions of the Fubini and Tonelli Theorems hold when $(X, \mathcal{M}, \mu)$ is an arbitrary measure space (not necessarily σ-finite) and Y is a countable set, $\mathcal{N} = \mathcal{P}(Y)$, and ν is counting measure.

Proof. We show first that Tonelli's theorem holds. That is, we want to show for $f \in L^+(X \times Y, \mathcal{M} \times \mathcal{N})$,

(1)
$$x \mapsto \int_Y f_x d\nu$$
is $\mathcal{M}$-measurable and
$$y \mapsto \int_X f^y d\mu$$
is $\mathcal{N}$-measurable,

(2)
$$\int_{X\times Y} fd(\mu \times \nu) = \int_X \left[\int_Y f_x d\nu(y)\right] d\mu(x) = \int_Y \left[\int_X f^y d\mu(x)\right] d\nu(y).$$

Let's first show it for characteristic functions. Let $f = \chi_E$, $E \in \mathcal{M} \times \mathcal{N}$. We wish to show that
$$x \mapsto \int_Y f(x,y)d\nu(y) = \nu(E_x)$$
is $\mathcal{M}$ measurable. Notice that we can write this as
$$\nu(E_x) = \sum_{y\in Y} \chi_{E^y}(x)\nu(\{y\}) = \sum_{y\in Y} \chi_{E^y}(x).$$
We have that E^y is a measurable set for each y, so χ_{E^y} is a measurable function for each y. Moreover, since Y is countable, we get that this is a countable sum of measurable functions, and so measurable (we have that finite sums of measurable functions are measurable, and limit of measurable functions are measurable, so the countable sum of measurable functions will be measurable). Hence, $\nu(E_x)$ is measurable.

Clearly, we will have
$$y \mapsto \mu(E^y).$$
is measurable, since $\mathcal{N} = \mathcal{P}(Y)$, and so every set is measurable. So (1) holds.

For (2), notice that we can write
$$E = \bigsqcup_{y\in Y}(E^y \times \{y\}),$$
which is a countable union. Using this and the fact that Y is countable, we get
$$\int_{X\times Y} fd(\mu\times\nu) = (\mu\times\nu)(E) = (\mu\times\nu)\left(\bigsqcup_{y\in Y}(E^y\times\{y\})\right) = \sum_{y\in Y}(\mu\times\nu)(E^y\times\{y\}) = \sum_{y\in Y}\mu(E^y)$$
$$= \sum_{y\in Y}\int_X \chi_{E^y}(x)d\mu(x) = \int_Y\left(\int_X f^y d\mu(x)\right)d\nu(y).$$
We see that **Theorem 2.15** gives us
$$\sum_{y\in Y}\int_X \chi_{E^y}(x)d\mu(x) = \int_X \sum_{y\in Y}\chi_{E^y}(x)d\mu(x) = \int_X\left(\int_Y f_x(y)d\nu(y)\right)d\mu(x).$$
Hence, we have (2).

We now follow the proof of **Theorem 2.37**. Since we have it for characteristic functions, we get that it holds for nonnegative simple functions. Take $f \in L^+(\mu\times\nu)$. Take $\psi_n \subset \mathrm{SF}^+$ such that

$\psi_n \nearrow f$ as in **Theorem 2.10**. Then MCT gives that the corresponding $g_n = (\psi_n)^y$ and $h_n = (\psi_n)_x$ converge to $g = f^y$ and $h = f_x$, and they are such that

$$\int_X f^y(x)d\mu(x) = \lim \int g_n(x)d\mu(x) = \lim \int \psi_n d(\mu \times \nu) = \int f d(\mu \times \nu),$$

$$\int_Y f_x(y)d\nu(y) = \lim \int h_n(y)d\nu(y) = \lim \int \psi_n d(\mu \times \nu) = \int f d(\mu \times \nu).$$

Hence, we have that Tonelli's theorem holds in this scenario.

Since Tonelli's theorem holds, we get that Fubini's theorem holds, since we can apply Tonelli to the positive and negative part of the real and imaginary parts and then use linearity. Hence, we have that it holds for $f \in L^1(\mu \times \nu)$.

□

Problem 35. Suppose $f, g \in \mathcal{L}^1(\mathbb{R}, \lambda)$.

(1) Show that $y \mapsto f(x-y)$ is measurable for all $x \in \mathbb{R}$ and in $\mathcal{L}^1(\mathbb{R}, \lambda)$ for a.e. $x \in \mathbb{R}$.

(2) Define the convolution of f and g by

$$(f * g)(x) := \int_{\mathbb{R}} f(x-y)g(y)d\lambda.$$

Show that $f * g \in \mathcal{L}^1(\mathbb{R}, \lambda)$.

(3) Show that $\mathcal{L}^1(\mathbb{R}, \lambda)$ is a commutative $\mathbb{C}$-algebra under $\cdot, +, *$.

(4) Show that

$$\int_{\mathbb{R}} |f * g| \leq \int_{\mathbb{R}} |f| \int_{\mathbb{R}} |g|,$$

i.e. $||\cdot||_1$ is submultiplicative.

Proof.

(1) Fix $x \in \mathbb{R}$. Then we have that $f(x-y) =: h(y) = f \circ t$, where t is the map $t(y) = x - y$. Since t is continuous for fixed x, we get that h is the composition of a (Borel) measurable function with a continuous function, and so is measurable (while f is not necessarily Borel, we can redefine it using **Proposition 2.12** to get that it's equal almost everywhere to a Borel measurable function and so we get that it's Borel measurable almost everywhere, which is sufficient for all that comes). Since the choice of x was arbitrary, we get that it applies for all $x \in \mathbb{R}$.

Remark. I realize that this part is a typo, but I prove the correct statement in (2) anyways, so I decided to leave it.

Using Property (7) from the class notes, we have

$$\int h(y)d\lambda = \int f(x-y)d\lambda = \int f(y)d\lambda < \infty,$$

so $h(y) = f(x-y) \in \mathcal{L}^1(\mathbb{R}, \lambda)$.

(2) We first need to show that $f(x-y)g(y) \in \mathcal{L}^1(\mathbb{R}, \lambda)$ for almost every x. We first assume $f, g \geq 0$. Then we have that

$$\int (f * g)(x)d\lambda(x) = \int \left(\int f(x-y)g(y)d\lambda(y) \right) d\lambda(x)$$

is such that Tonelli's theorem applies, since products of measurable functions are measurable and these are nonnegative. Thus, using the translation invariance of the Lebesgue (found on **pg. 74** of Folland), we have

$$\int \left(\int f(x-y)g(y)d\lambda(y) \right) d\lambda(x) = \int \left(\int f(x-y)g(y)d\lambda(x) \right) d\lambda(y) = \int g(y) \left(\int f(x-y)d\lambda(x) \right) d\lambda(y)$$

$$= \int g(y) \left(\int f(x)d\lambda(x) \right) d\lambda(y) = \left(\int g(y)d\lambda(y) \right) \left(\int f(x)d\lambda(x) \right) < \infty.$$

For general f, g integrable, this gives us

$$\int \left(\int |f(x-y)||g(y)|d\lambda(y) \right) d\lambda(x) = \int \left(\int |f(x-y)||g(y)|d\lambda(x) \right) d\lambda(y)$$

$$= \left(\int |f(x)|d\lambda(x) \right) \left(\int |g(x)|d\lambda(x) \right) < \infty,$$

and so $f(x-y)g(y) \in \mathcal{L}^1$ for a.e. $x \in \mathbb{R}$ by Tonelli/Fubini. Furthermore, we also see that $f * g \in \mathcal{L}^1$, since

$$|(f * g)(x)| = \left| \int f(x-y)g(y)d\lambda(y) \right| \leq \int |f(x-y)||g(y)|d\lambda(y)$$

for a.e. $x \in \mathbb{R}$, and so by **Proposition 2.22** we have

$$\int |(f * g)(x)|d\lambda(x) = \int \left| \int f(x-y)g(y)d\lambda(y) \right| d\lambda(x) \leq \int \left(\int |f(x-y)||g(y)|d\lambda(y) \right) d\lambda(x) < \infty$$

(3) We have that $\mathcal{L}^1$ is a Banach space from the recitation notes, hence a vector space, and so it suffices to show that it satisfies left and right distributivity, compatibility with scalars, and commutativity. Throughout, let $f, g, h \in \mathcal{L}^1$, $a, b \in \mathbb{C}$.

We first need to check left distributivity; we have

$$((f+g) * h)(x) = \int (f+g)(x-y)h(y)d\lambda = \int f(x-y)h(y)d\lambda + \int g(x-y)h(y)d\lambda$$

$$= (f * h)(x) + (g * h)(x).$$

We have right distributivity as well;

$$(f * (g+h))(x) = \int f(x-y)(g+h)(y)d\lambda = \int f(x-y)g(y)d\lambda + \int f(x-y)h(y)d\lambda$$

$$= (f * g)(x) + (f * h)(x).$$

Finally, we have compatibility with scalars;

$$(af) * (bg)(x) = \int af(x-y)bg(y)d\lambda = ab \int f(x-y)g(y)d\lambda = (ab)(f * g)(x).$$

Thus, we have that $\mathcal{L}^1$ is a $\mathbb{C}$-algebra. To get that it's commutative, we need to show $(f * g)(x) = (g * f)(x)$. But this is clear, since we have

$$(f * g)(x) = \int f(x-y)g(y)d\lambda(y),$$

and doing a change of variables with $u = x - y$, we get

$$\int f(u)g(x-u)d\lambda(u) = (g * f)(x).$$

Notice that the change of variable is valid by **Theorem 2.47**. So we get that $\mathcal{L}^1$ is a commutative $\mathbb{C}$-algebra.

(4) Notice that in (2) we established that

$$\int |f * g| d\lambda = \int \left| \int f(x-y)g(y) \right| d\lambda(y) d\lambda(x) \leq \int \left(\int |f(x-y)||g(y)| d\lambda(y) \right) d(\lambda(x))$$

$$= \int (|f| * |g|) d\lambda = \left(\int |f| d\lambda \right) \left(\int |g| d\lambda \right),$$

so we have the desired inequality.

□

Remark. Thomas O'Hare was a collaborator for this homework.

Problem 36. For $f \in \mathcal{L}^1(\lambda^n)$, let M be the Hardy-Littlewood maximal function

$$(Mf)(x) := \sup\left\{\frac{1}{\lambda^n(Q)}\int_Q |f| d\lambda^n \ : \ Q \in C(x)\right\}$$

where $C(x)$ is the set of all cubes of finite length which contain x. Define

$$f(x) := \begin{cases} \frac{1}{|x|(\ln(|x|)^2} \text{ if } |x| \leq \frac{1}{2} \\ 0 \text{ if } |x| > \frac{1}{2}. \end{cases}$$

Show that $f \in \mathcal{L}^1(\lambda^n)$ but $Mf \notin \mathcal{L}^1_{\text{loc}}$.

Proof. To see that $f \in \mathcal{L}^1(\lambda^n)$, we have

$$\int f d\lambda = \int_{-1/2}^{1/2} \frac{1}{|x|\ln(|x|)^2}dx = \int_0^{1/2}\frac{1}{x\ln(x)^2}dx + \int_{-1/2}^0 -\frac{1}{x\ln(-x)^2}dx.$$

If we need to careful about this, we can examine the first integral (and similarly for the second integral) as

$$\lim_{a\to 0^+}\int_a^{1/2}\frac{dx}{x\ln(x)^2} = \int_0^{1/2}\frac{dx}{x\ln(x)^2},$$

and use Riemann integral tricks to evaluate this.

For the first integral, let $u = \ln(x)$, then $du = (1/x)dx$, so we have

$$\int \frac{du}{u^2} = -\frac{1}{\ln(x)}\bigg|_0^{1/2} = \frac{1}{\ln(2)}.$$

Similarly, for the second integral, we have

$$-\int_{-1/2}^0 \frac{1}{x\ln(-x)^2 dx}.$$

let $u = \ln(-x)$, then $du = (-1/x)dx$, so

$$\int \frac{1}{u^2}du = -\frac{1}{\ln(-x)}\bigg|_{-1/2}^0 = \frac{1}{\ln(2)}.$$

Hence,

$$\int f d\lambda = \frac{2}{\ln(2)} < \infty,$$

so f is in $\mathcal{L}^1(\lambda)$.

To see that $Mf \notin \mathcal{L}^1_{\text{loc}}$, we need to show that for some bounded H we have

$$\int_H Mf d\lambda = \infty.$$

Take $H = [0, 1/4]$. Then we have

$$(Mf)(x) \geq \frac{1}{x}\int_0^x \frac{dt}{t\ln(t)^2} = -\frac{1}{x\ln(x)}.$$

Notice that on $[0, 1/4]$ we have

$$\int_0^{1/4} (Mf)(x)dx \geq \int_0^{1/4} -\frac{dx}{x\ln(x)} = -\ln(\ln(|x|))\Big|_0^{1/4} = \infty.$$

So Mf is not locally integrable. $\square$

Problem 37. Suppose $E \subset \mathbb{R}^n$ (not assumed to be Borel measurable) and let $\mathcal{C}$ be a family of cubes covering E such that

$$\sup\{l(Q) \ : \ Q \in \mathcal{C}\} < \infty.$$

Show there exists a sequence $(Q_k) \subset \mathcal{C}$ of disjoint cubes such that

$$\sum_{k=1}^{\infty} \lambda^n(Q_k) \geq 5^{-n}(\lambda^n)^*(E).$$

Remark. Follows the proof given in Wheeden and Zygmund, "Integral and Measure."

Proof. Let $t_1^* = \sup\{l(Q) \ : \ Q \in \mathcal{C}\}$. Then we can choose Q_1 such that $Q_1 > (1/2)t_1^*$. Write $\mathcal{C} = \mathcal{C}_2 \cup \mathcal{C}_2'$, where we have $\mathcal{C}_2$ is the collection of cubes which are disjoint from Q_1 and $\mathcal{C}_2'$ is the collection of cubes which intersect Q_1. Let $\widehat{Q_1}$ be the cube which is concenteric with Q_1 and has edge length $5l(Q_1)$. Hence, since $2l(Q_1) > t_1^*$, we have that every cube in $\mathcal{C}_2'$ is contained in $\widehat{Q_1}$.

Continue this algorithm, letting $t_j^* = \sup\{l(Q) \ : \ Q \in \mathcal{C}_j\}$, choosing $Q_j \in \mathcal{C}_j$ where $l(Q_j) > \frac{1}{2}t_j^*$, and we split $\mathcal{C}_j = \mathcal{C}_{j+1} \cup \mathcal{C}_{j+1}'$, where the former contains all the cubes disjoint from Q_j and the latter contains all the cubes which intersect Q_j. Notice that we have $t_j^* \geq t_{j+1}^*$ by construction, and moreover for each j, we have $Q_1, \ldots, Q_j$ are disjoint from eachother and every other cube in $\mathcal{C}_{j+1}$, and every cube in $\mathcal{C}_{j+1}'$ is contained in the cube $\widehat{Q_j}$. We have that the process terminates if $\mathcal{C}_{j+1}$ is empty.

We now break it up into cases. Consider the case where we have $\mathcal{C}_{j+1}$ is empty for some j. Since

$$\mathcal{C} = \mathcal{C}_{j+1} \cup \mathcal{C}_{j+1}' \cup \cdots \cup \mathcal{C}_2',$$

and E is covered by the cubes in $\mathcal{C}$, it follows that E is covered by the cubes in $\mathcal{C}_{j+1}' \cup \cdots \cup \mathcal{C}_2'$. Hence, since $\widehat{Q_j}$ contains all cubes in the respective collection, we get

$$(\lambda^n)^*(E) \leq \sum_{i=1}^{j} \lambda^n(\widehat{Q_i}) = 5^n \sum_{i=1}^{j} \lambda^n(Q_i).$$

Hence, we have

$$5^{-n}(\lambda^n)^*(E) \leq \sum_{i=1}^{j} \lambda^n(Q_i).$$

On the other hand, we have it does not terminate at some point. Since $t_1^* \geq t_2^* \geq \cdots$, we either have that there is a $\delta > 0$ such that $t_j^* \geq \delta$ for all j or $t_j^* \to 0$. In the first case, we have that $l(Q_j) \geq (1/2)\delta$ for all j, and so we get

$$\sum_{i=1}^{\infty} \lambda^n(Q_i) = \infty.$$

Hence, we win.

In the second case, we need to show that every cube in $\mathcal{C}$ is contained in $\bigcup_j \widehat{Q_j}$. This follows, since if a cube wasn't contained in this, we would have that there would be a cube Q such that it does not intersect any Q_j, and so $l(Q) \leq t_j^*$ for every j, and hence would have length 0, a contradiction. Hence, we have that

$$5^n \sum_{i=1}^{\infty} \lambda^n(Q_i) \geq (\lambda^n)^*(E),$$

as desired. □

Problem 38. Let (X, τ) be a topological space. A net $(x_i)_{i \in \Lambda}$ is called *universal* if, for every subset $Y \subset X$, (x_i) is either eventually in Y or eventually in Y^c.

(1) Show that every net has a universal subnet. (Optional)
(2) Show that (X, τ) is compact if and only if every universal net converges.

Remark. The solution to (1) was inspired by Howes' "Modern Analysis and Topology."

Proof. (1) Let $\mathcal{C} = \{A \subset X \ : \ x \text{ is eventually in } X\}$. Notice that $\mathcal{C}$ has the property that, if x is eventually in $A \in \mathcal{C}$, x is eventually in $B \in \mathcal{C}$, then we have that $A \cap B = \varnothing$, since there is an N such for all $n \geq N$, $x_n \in A \cap B$. Notice that this gives us that $\mathcal{C}$ has the finite intersection property from **Homework 1**; that is, if $\{A_i\}_{i=1}^n \subset \mathcal{C}$ is a finite collection of sets, we have

$$\bigcap_{i=1}^{n} A_i = \varnothing$$

by the above remark. Notice as well that x is frequently in every member of $\mathcal{C}$. Notice that $\mathcal{C}$ is a directed set; that is, it is equipped with the obvious binary relation $\subset$, and we have that, for every $A, B \in \mathcal{C}$, there is a $C \in \mathcal{C}$ (can take it to be X as a whole) such that $A \subset C$ and $B \subset C$.

Examine now $\mathcal{P}(X)$, and take $S \subset \mathcal{P}(X)$ to be the set of collections of sets which contain $\mathcal{C}$ and satisfy the finite intersection property and the fact that x is frequently in each member of the collection. We can order S via inclusion ($\subset$). If we take a chain $\{K_\alpha\}$ in S under this ordering, then we want to see that $\bigcup K_\alpha$ also has these properties. First, it's clear that $\mathcal{C}$ is in $\bigcup K_\alpha$, since it is contained in each K_α. Next, if $A, B \in \bigcup K_\alpha$, we have that there is an α_0 such that $A, B \in K_{\alpha_0}$, and so $A \cap B = \varnothing$. Finally, taking a set $A \in \bigcup K_\alpha$, we have that $A \in K_{\alpha_0}$ for some α_0, and so (x_i) is frequently in A. Hence, $\bigcup K_\alpha \in S$. We can then use Zorn's Lemma to get that there is a maximal collection containing $\mathcal{C}$ and which has the two properties. Call this maximal collection T.

Take $A \subset X$. We wish to show that $A \in T$ or $A^c \in T$. Assume for contradiction that $A \notin T$ and $A^c \notin T$. Since $A \notin T$, we must have that there is a $B \in T$ such that $A \cap B = \varnothing$. However, we have $B \subset A^c$, and so since $B \in T$ we have that x is frequently in A^c. Taking any $C \in T$, we notice that $C \cap B \subset C \cap A^c$ by assumption, and hence since $C \cap B \neq \varnothing$ we must have that $C \cap A^c \neq \varnothing$. Thus, $A^c \subset X$ is such that it has the desired two properties, and so we get a contradiction on the maximality of T. Hence, either $A \in T$ or $A^c \in T$.

Thus, x is frequently in each member of T, and we wish to construct a subnet such that it is eventually in every member of T. We can make a space $E = \{(i, A) \in \Lambda \times T \ : \ x_i \in T\}$. Using the finite intersection property, this is a directed set via the natural choice; if $(i_0, A_0), (i_1, A_1) \in E$, then $(i_0, A_0) \leq (i_1, A_1)$ if $i_0 \leq i_1$ and $A_1 \subset A_0$. Notice that for all (i_0, A_0), $(i_1, A_1) \in E$, we have that there is a (i_2, A_2) such that $(i_0, A_0) \leq (i_2, A_2)$ and $(i_1, A_1) \leq (i_2, A_2)$; this follows by the fact that $A_0 \cap A_1 \neq \varnothing$, $A_0 \cap A_1 \in T$, and so x is frequently in $A_0 \cap A_1$ as well. Define a function $f : E \to \Lambda$ by $f(i, A) = i$. We need to check that this gives us a subnet; that is, it's monotone and for every $i_1 \in \Lambda$, there exists a $(i_0, A_0) \in E$ such that $f(i_0, A_0) \geq i$ (Wikipedia calls this **cofinal in the image**). To see monotonicity, if $(i_0, A_0) \le (i_1, A_1)$, then we have $f(i_0, A_0) = i_0 \le i_1 = f(i_1, A_1)$. The cofinal property follows immediately from definition.

So we have that $\{y_{(i,A)} : A \in E\}$ is a subnet defined by $y_{(i,A)} = x_{f(i,A)}$. If we establish that it's eventually in each member of T, then by the remark earlier we see that since for all $A \subset X$, either $A \in T$ or $A^c \in T$, we have that the subnet is eventually in either A or A^c, and so is universal.

Take $A_0 \in T$. We have that (x_i) is frequently in A by one of our two assumptions on T. Hence, there exists a $i_0 \in \Lambda$ such that $x_{i_0} \in A_0$, and so $(i_0, A_0) \in E$. Taking $(i_1, A_1) \in E$ arbitrarily such that $(i_0, A_0) \le (i_1, A_1)$, we have that $i_0 \le i_1$ and $A_1 \subset A_0$, so $y_{(i_1,A_1)} = x_{i_1} \in A_1 \subset A_0$, and so the subnet is eventually in A_0. Hence, we have found a subnet which is either eventually in A or A^c for every $A \subset X$.

(2) ($\Longrightarrow$) If (X, τ) is compact, then by the theorem from the notes we have that this is equivalent to every net having a cluster point and also equivalent to every net having a convergent subnet. We need to show that every universal net converges. Let (x_i) be a universal net on (X, τ), and let $x \in X$ be a point at which it clusters. Then for every neighborhood V such that $x \in V$, we have that (x_i) is frequently in V. Since (x_i) is universal, we must have that it is eventually in either V or V^c. Since it is frequently in V, we cannot have that it's eventually in V^c, and so it must eventually be in V. Since this applies for all neighborhoods of x, we have that (x_i) converges to x.
($\Longleftarrow$) Assume that every universal net converges. By (1), every net has a universal subnet, and so we have that every net has a subnet which converges. By the equivalence established in the notes, this tells us that X is compact.

□

Problem 39. Suppose (X, τ) is a locally compact Hausdorff topological space, and suppose $K \subset X$ is a non-empty compact set.

(1) Suppose $K \subset U$, where K compact and U open. Show there is a continuous function $f : X \to [0, 1]$ such that $f|_K = 1$ and $f|_{U^c} = 0$ (in other words, prove the LCH version of Urysohn's Lemma).
(2) Suppose $f : K \to \mathbb{C}$ is continuous. Show there is a continuous function $F : X \to \mathbb{C}$ such that $F|_K = f$ (in other words, prove the LCH version of the Tietze Extension Theorem).

Proof.

(1) It's clear that a subset of a Hausdorff space is Hausdorff (if X Hausdorff, $A \subset X$, then for every $x \ne y$ in A we can find open subsets U, V in X such that $x \in U$, $y \in V$, $U \cap V = \varnothing$. In the relative topology, we have $U \cap A$, $V \cap A$ are open, they are still disjoint, and $x \in U \cap A$, $y \in V \cap A$). Moreover, being locally compact means that for all $x \in X$, there is an open U such that $x \in U$ and $\overline{U}$ is compact.
Step 1: For each $x \in K$, we would like to find U_x such that $x \in U_x \subset \overline{U_x} \subset U$. Notice that we can find $x \in V_x$ open where $\overline{V_x}$ is compact. Since $\overline{U}$ may not be compact, we set $V := U \cap V_x \subset U$. Notice that $\overline{V}$ is compact and V open. We have $\overline{V} - V$ is a closed set of $\overline{V}$, and since $\overline{V}$ is compact this tells us that $\overline{V} - V$ is compact. For each $y \in \overline{V} - V$, we can find disjoint open sets $W_y, E_y \subset \overline{V}$ such that $x \in W_y$, $y \in E_y$. Since $\overline{V} - V$ compact, $\overline{V} - V \subset \bigcup_{y \in \overline{V} - V} E_y$, we can find a finite subcover, $\overline{V} - V \subset \bigcup_{i=1}^n E_i$. Corresponding to each E_i is a W_i, and so setting $W = \bigcap_{i=1}^n W_i$, $E = \bigcup_{i=1}^n E_i$, we have $W \cap E = \varnothing$, W and E are open relative to $\overline{V}$, and $x \in W$, $\overline{V} - V \subset E$. Notice as well

$$W \subset \overline{V} - E \subset \overline{V} - (\overline{V} - V) = \overline{V} \cap (\overline{V} \cap V^c)^c = \overline{V} \cap (\overline{V}^c \cup V) = \overline{V} \cap V = V,$$

so that W is open in X. Moreover, since $\overline{V} \cap E^c$ is closed,

$$\overline{W} \subset \overline{V} - E \subset V,$$

hence $\overline{W}$ is compact subset of V. So letting $U_x = W$, we have that for each $x \in K$, we can find a U_x such that

$$K \subset U_x \subset \overline{U_x} \subset U.$$

Step 2: By **Step 1**, we have that

$$K \subset \bigcup_{x \in U} U_x \subset U,$$

where U_x is such that $K \subset U_x \subset \overline{U_x} \subset U$. Since K is compact, we can choose a finite subcover; that is, we have

$$K \subset \bigcup_{i=1}^{n} U_i \subset U,$$

where U_i is such that $\overline{U_i}$ is compact. Set $V := \bigcup_{i=1}^{n} U_i$. Notice that $\overline{V} = \bigcup_{i=1}^{n} \overline{U_i}$. So we have $K \subset V \subset \overline{V} \subset U$, V is open and $\overline{V}$ is compact. Since $\overline{V}$ is closed and compact, as well as Hausdorff, we have that it's normal. Furthermore, K compact implies it's closed, and $K \cap (\overline{V} \cap V^c) = \varnothing$, so Urysohn's Lemma from the notes applies to give us a $f \in C(\overline{V}, [0,1])$ such that $f|_K = 1$ and $f|_{\overline{V} \cap V^c} = 0$. We set $f = 0$ on $\overline{V}^c$ to extend it to all of X.

We now need to show that f is continuous. Take $E \subset [0,1]$ closed. If $0 \in E$, we get that $f^{-1}(E) = (f|_{\overline{V}})^{-1}(E) \cup V^c$. Since V open, V^c closed, and since f continuous on $\overline{V}$ we have that $(f|_{\overline{V}})^{-1}(E)$ is closed as well, and finite unions of closed sets are closed. Hence, in this case, it pulls back closed sets to closed sets. If $0 \notin E$, we have that $f^{-1}(E) = (f|_{\overline{V}})^{-1}(E)$, which is closed since f is continuous on this domain. Hence, it pulls back closed sets to closed sets, and so is continuous.

(2) From (1), we have that there is a U open such that $K \subset U \subset \overline{U}$, and $\overline{U}$ is compact. Hence, $\overline{U}$ is a normal space. We can use the Tietze Extension Theorem from the notes in this case, noting that K is closed, to find a $F \in C(\overline{U}, \mathbb{C})$ where $F|_K = f$. We now use a bump function type of argument to extend this to the whole space. We can find $g : X \to [0,1]$ continuous such that $g|_K = 1$ and $g|_{U^c} = 0$ by (1) (that is, the LCH Urysohn's Lemma). Define

$$\widehat{F}(x) := \begin{cases} F(x)g(x) \text{ if } x \in \overline{U} \\ 0 \text{ otherwise.} \end{cases}$$

We would like to then check that this is continuous. Notice that a product of continuous functions is continuous, and so $\widehat{F}$ is continuous on $\overline{U}$. Take $E \subset \mathbb{C}$ closed. We have

$$\widehat{F}^{-1}(E) = \widehat{F}^{-1}([E \cap \{0\}] \sqcup [E \cap \{0\}^c]) = \widehat{F}^{-1}(E \cap \{0\}) \sqcup \widehat{F}^{-1}(E \cap \{0\}^c).$$

Notice that

$$\widehat{F}^{-1}(E \cap \{0\}) = \begin{cases} (F \cdot g)^{-1}(\{0\}) \text{ if } E \cap \{0\} \neq \varnothing \\ \varnothing \text{ otherwise} \end{cases}$$

$$\widehat{F}^{-1}(E - \{0\}) = \widehat{F}^{-1}(E) \cap \overline{U} = \left(\widehat{F}|_{\overline{U}}\right)^{-1}(E),$$

so the preimage is a union of two closed sets, which is closed. Hence, $\widehat{F}$ is continuous, and is such that $\widehat{F}|_K$ is f.

□

Problem 40. Suppose (X, τ) is a locally compact topological space and (f_n) is a sequence of continuous $\mathbb{C}$-valued functions on X. Show that the following are equivalent:

(1) There is a continuous function $f : X \to \mathbb{C}$ such that $f_n|_K \to f|_K$ uniformly on every compact $K \subset X$.
(2) For every compact $K \subset X$, $(f_n|_K)$ is uniformly Cauchy.

Proof. (1) $\implies$ (2): Let $K \subset X$ be an arbitrary compact set. Notice that this implies that for all $\epsilon > 0$, we have that there is an N such that for all $n \geq N$, $x \in K$

$$|f_n(x) - f(x)| < \epsilon.$$

Fix $\epsilon > 0$. Then we have that we can find an N such that for all $n, m \geq N$, $x \in K$

$$|f_n(x) - f(x)| < \frac{\epsilon}{2}, \quad |f_m(x) - f(x)| < \frac{\epsilon}{2}.$$

Notice that this then implies

$$|f_n(x) - f_m(x)| = |f_n(x) - f(x) + f(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence, for all $\epsilon > 0$, we can find an N such that for all $n, m \geq N$, $x \in K$, we have

$$|f_n(x) - f_m(x)| < \epsilon.$$

(2) $\implies$ (1): Define $f(x) := \lim_{n\to\infty} f_n(x)$. We see this is well-defined, since we can take a compact set K around x and use the fact that $(f_n|_K)$ is uniformly Cauchy on this. Furthermore, it's clear that $f_n|_K \to f|_K$ uniformly on every compact $K \subset X$; given some compact subset $K \subset X$, $\epsilon > 0$, we have that there is an N such that for all $n, m \geq N$, for all $x \in K$,

$$|f_n(x) - f_m(x)| < \epsilon,$$

and so since the norm is continuous we can take the limit as $n \to \infty$ to get

$$|f(x) - f_m(x)| \leq \epsilon.$$

Since this works for all $x \in K$, all $\epsilon > 0$, we have uniform convergence. It is a uniform limit of continuous functions, and so we have that f is continuous on every compact set $K \subset X$.

It remains to show that f is continuous on all of X. That is, we need to show that

$$\lim_{n\to\infty} f(x_n) = f(x),$$

where $x = \lim_{n\to\infty} x_n$. Take the set

$$E := \{x_n \ : \ n \in \mathbb{N}\} \cup \{x\}.$$

Since X is locally compact, we have that there is an open U such that $x \in U$ and $\overline{U}$ is compact. Since $x_n \to x$, we have that there is an N such that for all $n \geq N$, $x_n \in U \subset \overline{U}$. Since $\overline{U}$ is compact, we have that f is continuous on $\overline{U}$, and so we have that $\lim_{n\to\infty} f(x_n) = f(x)$, as desired. $\square$

Remark. Thomas O'Hare was a collaborator.

Problem 41. A filter on a set X is a collection $\mathcal{F}$ of non-empty subsets of X satisfying

(1) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$, and
(2) $A \in \mathcal{F}$ and $A \subset B$ implies $B \in \mathcal{F}$.

Suppose τ is a topology on X. We say a filter *converges* to $x \in X$ if every open neighborhood U of x lies in $\mathcal{F}$.

(1) Show that $A \subset X$ is open if and only if $A \in \mathcal{F}$ for every filter $\mathcal{F}$ that converges to a point in A.
(2) Show that $\mathcal{F}$ and $\mathcal{G}$ are filters and $\mathcal{F} \subset \mathcal{G}$ ($\mathcal{G}$ is a *subfilter* of $\mathcal{F}$), then $\mathcal{G}$ converges to x whenever $\mathcal{F}$ converges to x.
(3) Suppose (x_λ) is a net in X. Let $\mathcal{F}$ be the collection of sets A such that (x_λ) is eventually in A. Show that $\mathcal{F}$ is a filter. Then show that $x_\lambda \to x$ if and only if $\mathcal{F}$ converges to x.

Proof. (1) ($\implies$) Assume $A \subset X$ is open. Let $\mathcal{F}$ be a filter which converges to x in A. Then we have that every open neighborhood of x lies in $\mathcal{F}$, and A is an open neighborhood of x, so $A \in \mathcal{F}$.

($\impliedby$) For every point $x \in A$, take an open neighborhood U_x such that $U_x \subset A$. This is possible, since $A \in \mathcal{F}$, the filter which converges to x, and since every open neighborhood of x is in $\mathcal{F}$ the only way that $A \in \mathcal{F}$ is if there is an open neighborhood contained in A. Since we can do this for all x, we have

$$\bigcup_{x \in A} U_x \subset A \subset \bigcup_{x \in A} U_x,$$

or in other words,

$$\bigcup_{x \in A} U_x = A.$$

Hence, A is open.

(2) If every open neighborhood of x is in $\mathcal{F}$, then this implies that every open neighborhood of x is in $\mathcal{G}$, and so $\mathcal{G}$ converges to x whenever $\mathcal{F}$ converges to x.

(3) We first show that $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$. If $A \in \mathcal{F}$, this implies there is a point λ_1 such that, for all $t \geq \lambda_1$, we have $x_t \in A$. Likewise, $B \in \mathcal{F}$ implies that there is a λ_2 such that, for all $t \geq \lambda_2$, we have $x_t \in B$. By assumption, we can find λ' such that $\lambda' \geq \lambda_1$ and $\lambda' \geq \lambda_2$. Hence, for all $t \geq \lambda'$, we have that $x_t \in A \cap B$. So x is eventually in $A \cap B$.

Now, if $A \in \mathcal{F}$, and $A \subset B$, then clearly if (x_λ) is eventually in A, it is eventually in B, so $B \in \mathcal{F}$. Finally, $\mathcal{F}$ is non-empty, since (x_λ) is in X, so $X \in \mathcal{F}$.

We now show the if and only if statement.

($\implies$) If $x_\lambda \to x$, for all neighborhoods U of x, we have that x_λ is eventually in U, so this in particular holds for all open neighborhoods, and so all open neighborhoods are in $\mathcal{F}$. Hence, $\mathcal{F}$ converges to x.

($\impliedby$) If $\mathcal{F}$ converges to x, then for all open neighborhoods U, we have that (x_λ) is eventually in U. For every neighborhood, we have that we can take the interior to get an open neighborhood about the point, and so (x_λ) is eventually in every neighborhood of x, and so (x_λ) converges to x.

□

Problem 42. A filter $\mathcal{F}$ is called an *ultrafilter* if it is not properly contained in any other filter.

(1) Show that a filter $\mathcal{F}$ is an ultrafilter if and only if for every $A \subset X$, we have either $A \in F$ or $A^c \in \mathcal{F}$.
(2) Use Zorn's lemma to prove that every filter is contained in an ultrafilter.

Proof. (1) ($\implies$) Let $\mathcal{F}$ be an ultrafilter, $A \subset X$. Then if $A \in \mathcal{F}$, we win. Otherwise, assume $A \notin F$. We wish to show that $A^c \in \mathcal{F}$. If $A^c \notin F$, then we can create a bigger filter as follows: let $\mathcal{G}$ be the filter generated by $\mathcal{F}$ and A^c. That is, $\mathcal{F} \subset \mathcal{G}$, $A^c \in \mathcal{G}$, and every set containing $A^c \in \mathcal{G}$, $\mathcal{G}$ contains $B \cap E$ for all $E \in \mathcal{F}$, $A^c \subset B$, and $\mathcal{G}$ contains all subsets which contain $B \cap E$ for $E \in \mathcal{F}$, $A^c \subset B$. This is clearly a filter; it's nonempty since $\mathcal{F} \subset \mathcal{G}$, we have that it's closed under finite intersections by construction, and we have that if $E \in \mathcal{G}$, $E \subset F$, then by construction $F \in \mathcal{G}$. Hence, $\mathcal{F} \subset \mathcal{G}$ properly, but this contradicts $\mathcal{F}$ being an ultrafilter, and so we must have had $A^c \in \mathcal{F}$.
($\impliedby$) Let $\mathcal{F}$ be the filter which for every $A \subset X$, either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$. Let $\mathcal{G}$ be such that $\mathcal{F} \subset \mathcal{G}$. We wish to show that $\mathcal{G} = \mathcal{F}$. Assume for contradiction it did not; we have that there is some $L \in \mathcal{G}$ such that $L \notin \mathcal{F}$. This implies that $L^c \in \mathcal{F}$. However, this then gives us that $L, L^c \in \mathcal{G}$, so $\varnothing \in \mathcal{G}$, which contradicts $\mathcal{G}$ being a filter. Hence, $\mathcal{F}$ is not properly contained in any filter.
(2) We need to create a chain of filters. Let $\mathcal{F}$ be a filter on X. If $\mathcal{F}$ is an ultrafilter, we are done. Otherwise, let $\mathcal{F}_0 = \mathcal{F}$, and since $\mathcal{F}$ is not an ultrafilter, it must be properly contained in some other filter, say $\mathcal{F}_1$. Continue this process, creating a chain of filters;

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots .$$

We then want to establish that

$$\mathcal{F} := \bigcup_i \mathcal{F}_i$$

is also an filter; that is, the chain has an upper bound which is a filter. First, notice that $\mathcal{F}$ is non-trivial, since all the $\mathcal{F}_i$ are, and furthermore doesn't contain any empty subsets of X. We then show the first condition. Take $A, B \in \mathcal{F}$. Then $A \in \mathcal{F}_n$, $B \in \mathcal{F}_m$. Without loss of generality, assume $n \geq m$. Then $A, B \in \mathcal{F}_n$, and we have that $A \cap B \in \mathcal{F}_n \subset \mathcal{F}$. So the first condition holds.

Next, take $A \in \mathcal{F}$. Then we have $A \in \mathcal{F}_n$ for some n. If B is such that $A \subset B$, then $B \in \mathcal{F}_n \subset \mathcal{F}$. So the second condition holds as well. Hence, $\mathcal{F}$ is a filter. So in the space of filters on X, every chain has an upper bound which is a filter, say $\mathcal{F}$. Notice that this upper bound is such that there is no filter $\mathcal{G}$ which properly contains $\mathcal{F}$, since this contradicts the maximality of $\mathcal{F}$. Since we can do this process for any choice of filter $\mathcal{G}$, we have that every filter is contained in an ultrafilter.

□

Problem 43. Show that the following collections of functions are uniformly dense in the appropriate algebras:

(1) For $a < b$ in $\mathbb{R}$, the polynomials $\mathbb{R}[t] \subset C([a, b], \mathbb{R})$.
(2) For $a < b$ in $\mathbb{R}$, the piecewise linear functions $PWL \subset C([a, b], \mathbb{R})$.
(3) For $K \subset \mathbb{C}$ compact, the polynomials $\mathbb{C}[z] \subset C(K)$.
(4) For $\mathbb{R}/\mathbb{Z}$, the trigonometric polynomials span

$$\{\sin(2\pi n x), \cos(2\pi n x) \ : \ n \in \mathbb{N} \cup \{0\}\} \subset C(\mathbb{R}/\mathbb{Z}, \mathbb{R}).$$

Remark. We need the following claim throughout (wasn't sure if we need this, since we covered something similar in recitation, but I'm leaving it in anyways).

Claim. Let $\mathcal{A}$ be a subalgebra of either $C(X, \mathbb{R})$ or $C(X, \mathbb{C})$. The closure of $\mathcal{A}$, denoted $\overline{\mathcal{A}}$, is still a subalgebra.

Proof. Let $F = \mathbb{R}, \mathbb{C}$, $\mathcal{B} = \overline{\mathcal{A}}$. We need to show that for all $\alpha, \beta \in F$, $f, g \in \mathcal{B}$, $\alpha f + \beta g \in \mathcal{B}$. Notice first that $f \in \mathcal{B}$ implies there is a sequence $(f_n) \subset \mathcal{A}$ such that $f_n \to f$. Since $\alpha f_n \in \mathcal{A}$ for all n, we get $\alpha f_n \to \alpha f \in \mathcal{B}$. So it really suffices to show that for all $f, g \in \mathcal{B}$, we have $f + g \in \mathcal{B}$. Notice that $f, g \in \mathcal{B}$ implies there are $(f_n) \subset \mathcal{A}$, $(g_n) \subset \mathcal{A}$ such that $f_n \to f$, $g_n \to g$. Since addition is continuous and $f_n + g_n \in \mathcal{A}$ for all n, we get $f_n + g_n \to f + g$. Finally, we need to show that for all $f, g \in \mathcal{B}$, $fg \in \mathcal{B}$. Let $(f_n) \subset \mathcal{A}$, $(g_n) \subset \mathcal{A}$ be such that $f_n \to f$, $g_n \to g$. Notice that

$$f_n g_n - fg = f_n(g_n - g) + g(f_n - f).$$

So we have that

$$||f_n g_n - fg|| \leq ||f_n|| \cdot ||g_n - g|| + ||g|| \cdot ||f_n - f||.$$

Notice that

$$f_n = f + (f_n - f),$$

and so we have

$$||f_n|| \leq ||f|| + ||f_n - f||.$$

Since $f_n \to f$, we can choose n sufficiently large so that $||f_n - f|| \leq 1$, so that

$$||f_n|| \leq 1 + ||f||.$$

Using this, we have

$$||f_n g_n - fg|| \leq (1 + ||f||) \cdot ||g_n - g|| + ||g|| \cdot ||f_n - f|| \to 0.$$

Hence, we get that $f_n g_n \to fg$, and so $fg \in \mathcal{B}$. Thus, $\mathcal{B}$ is a subalgebra. ☐

Proof.

(1) We apply the Stone-Weierstrass theorem. Notice that $\mathbb{R}[t]$ separates points in $[a, b]$ by taking $p(x) = x \in \mathbb{R}[t]$. Then if $x \neq y$, we have $p(x) = x \neq y = p(y)$. We then need to check that $\mathbb{R}[t]$ is a subalgebra of $C([a, b], \mathbb{R})$. Notice that if $f, g \in \mathbb{R}[t]$, then $fg \in \mathbb{R}[t]$ (the product of polynomials is a polynomial), and so it suffices to check it is a vector subspace of $C(X, \mathbb{R})$. Notice that, for $\alpha, \beta \in \mathbb{R}$, $f, g \in \mathbb{R}[t]$, we have $\alpha f + \beta g \in \mathbb{R}[t]$, since scaling a polynomial is still a polynomial and the sum of polynomials is a polynomial. Hence, the theorem tells us that $\overline{\mathbb{R}[t]} = C([a, b], \mathbb{R})$, since the constant function is a polynomial, and so we have that it is uniformly dense.

(2) Let $p(x) \in C([a, b], \mathbb{R})$ be some arbitrary continuous function. Partition the interval $[a, b]$ using the uniform continuity of continuous functions on compact intervals. That is, fix $\epsilon > 0$. Then since we are uniformly continuous on $[a, b]$, we can find δ such that $|x - y| < \delta$ implies $|p(x) - p(y)| < \epsilon/2$. Divide up the interval into n subintervals such that $(b - a)/n < \delta$. Say we have $a = x_0 < x_1 < \cdots < x_n = b$. Define f to be the piecewise linear function which is equal to $f(x_i) = p(x_i)$ for each $i = 1, \ldots, n$, and where for $x \in (x_i, x_{i+1})$ we have f is the line connecting these two points. Then we have that for each $x \in [a, b]$, $x \in [x_i, x_{i+1})$ for some $i = 1, \ldots, n - 1$ or $x \in [x_{n-1}, x_n]$. Let x_i denote the minimum value on the interval. Thus, we have

$$|p(x) - f(x)| = |p(x) - f(x_i) + f(x_i) - f(x)| \leq |p(x) - f(x_i)| + |f(x_i) - f(x)|.$$

Using the fact that $f(x_i) = p(x_i)$, we have that this is equal to

$$\begin{aligned} |p(x) - p(x_i)| + |f(x_i) - f(x)| &\leq |p(x_{i+1}) - p(x_i)| + |f(x_i) - f(x)| \\ &= |f(x_{i+1}) - f(x_i)| + |f(x_i) - f(x)|. \end{aligned}$$

Now, since $|x_{i+1} - x_i| < \delta$ by construction, we have that this is bounded above by $\epsilon/2 + \epsilon/2 = \epsilon$. Since this applied for all $x \in [a, b]$, we have that

$$||f - p|| < \epsilon.$$

So $\overline{\text{PWL}} = C([a,b],\mathbb{R})$, since we can approximate functions in $C([a,b],\mathbb{R})$ arbitrarily well under the uniform norm.

(3) As stated, this problem is not true. Remediating it by throwing in $\overline{x}$, we have that it does hold. To see this, first notice that $\mathbb{C}$ is Hausdorff, so K is a compact Hausdorff space, and we're examining

$$\mathbb{C}[x,\overline{x}] \subset C(K,\mathbb{C}).$$

We need to check that $\mathbb{C}[x,\overline{x}]$ separates points, it is a subalgebra, and is closed under complex conjugation. The identity function $p(x) = x$ is in $\mathbb{C}[x,\overline{x}]$, so we have that it separates points. Next, it's a subalgebra, since for all $\alpha,\beta \in \mathbb{C}$, $f,g \in \mathbb{C}[x]$, we have $\alpha f + \beta g \in \mathbb{C}[x]$, and furthermore $fg \in \mathbb{C}[x]$ (product of polynomials is a polynomial, scaling and adding polynomials still gives a polynomial). Finally, we need to show that it is closed under complex conjugation, but this is clear since we've thrown in $\overline{x}$. Hence, Stone-Weierstrass applies, and we have that it is uniformly dense since the constant functions are in $\mathbb{C}[x,\overline{x}]$.

(4) We can interpret this to say that the span of the trigonometric polynomials are dense in the real-valued functions on $\mathbb{R}$ that are periodic with period 2π (the way it's written is equivalent, just changing the period to be 1 instead of 2π. I did it this way so I could use my old Fourier notes). Identifying it this way, we have that it's a continuous function $f : S^1 \cong [0,2\pi)/\sim \to \mathbb{R}$, and we note that S^1 is compact and Hausdorff under the subspace topology given by $\mathbb{R}^2$. Recall that trig polynomials are functions of the form

$$c_0 + \sum_{k=1}^{n} \left[a_k \cos(kx) + b_k \sin(kx)\right],$$

$c_0, a_i, b_i \in \mathbb{R}$. Let $\mathcal{A}$ be the space which is the span of all trig polynomials. It's clear that this is a vector subspace, since for $\alpha,\beta \in \mathbb{R}$, g,h trig polynomials, we get $\alpha g + \beta h \in \mathcal{A}$. Next we want to show that products of trig polynomials are trig polynomials. Once we show that $\cos(kx)\sin(lx)$, $\cos(kx)\cos(lx)$, $\sin(kx)\sin(lx)$ are trig polynomials, we are done by distributivity. To do this, we use the fact that

$$e^{ix} = \cos(x) + i\sin(x).$$

Hence,

$$e^{-ix} = \cos(x) - i\sin(x),$$

$$e^{ix} + e^{-ix} = 2\cos(x) \leftrightarrow \frac{1}{2}(e^{ix} + e^{-ix}) = \cos(x).$$

Similarly,

$$\frac{1}{2i}(e^{ix} - e^{-ix}) = \sin(x).$$

So

$$\cos(kx)\sin(lx) = \left(\frac{e^{ikx} + e^{-ikx}}{2}\right)\left(\frac{e^{ilx} - e^{-ilx}}{2i}\right)$$

$$= \frac{e^{i(k+l)x} - e^{i(k-l)x} + e^{-i(k-l)x} - e^{-i(k+l)x}}{4i} = \frac{1}{2}\sin((k+l)x) - \frac{1}{2}\sin((k-l)x).$$

So this is a trigonometric polynomial still. Analogously, we have

$$\cos(kx)\cos(lx) = \left(\frac{e^{ikx} + e^{-ikx}}{2}\right)\left(\frac{e^{ilx} + e^{-ilx}}{2}\right)$$

$$= \frac{e^{i(k+l)x} + e^{i(k-l)x} + e^{-i(k-l)x} + e^{-i(k+l)x}}{4} = \frac{1}{2}\cos((k+l)x) + \frac{1}{2}\cos((k-l)x),$$

$$\sin(kx)\sin(lx) = \left(\frac{e^{ikx}-e^{-ikx}}{2i}\right)\left(\frac{e^{ilx}-e^{-ilx}}{2i}\right)$$
$$= \frac{-e^{i(k+l)x}+e^{i(k-l)x}+e^{-i(k-l)x}-e^{-i(k+l)x}}{4} = -\frac{1}{2}\cos((k+l)x)+\frac{1}{2}\cos((k-l)x).$$
Hence, these are also trigonometric polynomials, so the product of two trigonometric polynomials is still a trigonometric polynomial. Hence, it's a subalgebra. Next, we need to establish that it separates points. That is, for $x, y \in S^1$, $x \neq y$ we need to show that there is a trigonometric polynomial where $f(x) \neq f(y)$. Notice that sin and cos are projections onto the x and y axis, and so we have that if $\sin(x) = \sin(y)$ while $x \neq y$, this means that $\cos(x) \neq \cos(y)$, and vice versa. Hence, have that the algebra separates points. Finally, it's clear that constant functions are in this algebra, and so we have that $\overline{\mathcal{A}} = C(S^1, \mathbb{R})$.

□

Problem 44. Let X, Y be compact Hausdorff spaces. For $f \in C(X)$, $g \in C(Y)$, define
$$(f \otimes g)(x, y) := f(x)g(y).$$
Prove that $\operatorname{span}\{f \otimes g \ : \ f \in C(X), g \in C(Y)\}$ is uniformly dense in $C(X \times Y)$.

Proof. We apply Stone-Weierstrass. First, denote $\operatorname{span}\{f \otimes g \ : \ f \in C(X), g \in C(Y)\} = C(X) \otimes C(Y)$. Then since the constant functions are in $C(X), C(Y)$, we get that they are in $C(X) \otimes C(Y)$, and so this does not vanish entirely on any point. Next, let $(x, y) \neq (a, b)$. Then we can find $f \in C(X)$ such that $f(x) \neq f(a)$, $g \in C(Y)$ such that $g(y) \neq g(b)$, and so $f(x)g(y) \neq f(a)g(b)$. Hence, $C(X) \otimes C(Y)$ separates points. Since $C(X)$, $C(Y)$ are closed under complex conjugation, we see that $C(X) \otimes C(Y)$ is as well; taking $h \in C(X) \otimes C(Y)$, we have
$$\overline{h}(x) = \overline{\left(\sum_{i=1}^{n} f(x_i)g(y_i)\right)} = \sum_{i=1}^{n} \overline{f(x_i)}\ \overline{g(y_i)} \in C(X) \otimes C(Y).$$
Moreover, we have that the product of continuous functions is continuous, and so $C(X) \otimes C(Y)$ is closed under multiplication, and is clearly a vector subspace, so a subalgebra of $C(X \times Y)$. Hence, the Stone-Weierstrass theorem tells us that
$$\overline{C(X) \otimes C(Y)} = C(X \times Y).$$
So it's uniformly dense. □

Problem 45. Suppose X is LCH and noncompact, and $A \subset C_0(X, \mathbb{C})$ is a subalgebra which separates points and is closed under complex conjugation. Show that either:

(1) $\overline{A} = C_0(X, \mathbb{C})$
(2) there is a $x_0 \in X$ such that $\overline{A} = \{f \in C_0(X, \mathbb{C}) \ : \ f(x_0) = 0\}$.

Proof. We follow the hint outlined in Folland (and in the lecture notes). First, assume that there is no $x \in X$ such that $f(x) = 0$ for all $f \in \mathcal{A}$. Then we let Y be the one-point compactification of X, taking the point to be ∞. We have that there is a unique extension of $f \in C_0(X, \mathbb{C})$ to $\widetilde{f} \in C_0(Y, \mathbb{C})$ via taking
$$\widetilde{f}(x) = \begin{cases} 0 \text{ if } x = \infty, \\ f(x) \text{ otherwise.} \end{cases}$$
Notice that this is unique, since if there were another $g \in C_0(Y, \mathbb{C})$ where $g(x) = f(x)$ for all $x \in X$, then the only thing that can happen is if $g(x) = f(x) + c$, where c is a constant (see **Proposition 4.36**), and this forces $c = 0$. We can analogously extend all of the functions in $\mathcal{A}$ in the same way, and so we get $\mathcal{A}' = \{\widetilde{f} \ : \ \widetilde{f}|_X \in \mathcal{A}\}$. Since $\mathcal{A}$ is a subalgebra, closed under complex

conjugation, and separates points, $\mathcal{A}'$ is a subalgebra, is closed under complex conjugation, and separates points. So we have $\overline{\mathcal{A}'}$ satisfies the criteria for Stone-Weierstrass, and so we apply it to get $\overline{\mathcal{A}'} = \{f \in C(Y, \mathbb{C}) \ : \ f(\infty) = 0\}$. By earlier, this is equal to the unique extension of $C_0(X, \mathbb{C})$ to $C(Y, \mathbb{C})$, and so we must have that $\overline{\mathcal{A}} = C_0(X, \mathbb{C})$.

Now, assume there is an $x_0 \in X$ such that $f(x_0) = 0$ for all $f \in \mathcal{A}$. We look at the space $X_0 = X - \{x_0\}$, and take the one-point compactification Y of X_0. We can again uniquely extend all of the functions, and again we get that $\overline{\mathcal{A}'} = \{f \in C(Y, \mathbb{C}) \ : \ f(\infty) = 0\}$, and so we have that this means that $\overline{\mathcal{A}'} = C_0(X_0, \mathbb{C})$. Hence, we have $\overline{\mathcal{A}'} = \{f \in C_0(X, \mathbb{C}) \ : \ f(x_0) = 0\}$. $\square$

Remark. Thomas O'Hare was a collaborator.

Problem 46. Let $U\mathbb{N}$ be the set of ultrafilters on $\mathbb{N}$. For a subset $S \subset \mathbb{N}$, define

$$[S] := \{\mathcal{F} \in U\mathbb{N} \ : \ S \in \mathcal{F}\}.$$

Show that the function $S \mapsto [S]$ satisfies the following properties:

(1) $[\varnothing] = \varnothing$ and $[\mathbb{N}] = U\mathbb{N}$.
(2) For all $S, T \subset \mathbb{N}$,
  (a) $[S] \subset [T]$ if and only if $S \subset T$.
  (b) $[S] = [T]$ if and only if $S = T$.
  (c) $[S] \cup [T] = [S \cup T]$.
  (d) $[S] \cap [T] = [S \cap T]$.
  (e) $[S^c] = [S]^c$.
(3) Find a sequence of subsets (S_n) of $\mathbb{N}$ such that $[\bigcup S_n] \neq \bigcup [S_n]$.
(4) Find a sequence of subsets (S_n) of $\mathbb{N}$ such that $[\bigcap S_n] \neq \bigcap [S_n]$.

Proof. (1) No filter contains the empty set, and so $[\varnothing] = \varnothing$. Similarly, every filter must contain $\mathbb{N}$, so $[\mathbb{N}] = U\mathbb{N}$.

(2) Let $S, T \subset \mathbb{N}$.
  (a) ($\implies$) Assume $S \not\subset T$. Construct a filter of sets which contain S. Then this is a filter which does not contain T. In particular, we can append T^c and all sets which contain T^c and $S \cap T^c$, and take the ultrafilter containing this filter (i.e. a filter $\mathcal{F} \in [S \cap T^c]$). Then we have an ultrafilter such that $S \in \mathcal{F}$ but $T \notin \mathcal{F}$, and so $[S] \not\subset [T]$.
  ($\impliedby$) If $S \subset T$, then, in particular, any filter $\mathcal{F}$ which contains S will contain T, and so we get $[S] \subset [T]$.
  (b) ($\implies$) If $[S] = [T]$, we have $[S] \subset [T]$, so $S \subset T$, and $[T] \subset [S]$, so $T \subset S$, and therefore $S = T$.
  ($\impliedby$) Similarly, if $S = T$, we have that $S \subset T$, so $[S] \subset [T]$, and $T \subset S$, so $[T] \subset [S]$. Hence, $[S] = [T]$.
  (c) Notice that $S \subset S \cup T$, $T \subset S \cup T$, so we have $[S] \subset [S \cup T]$, $[T] \subset [S \cup T]$, and hence $[S] \cup [T] \subset [S \cup T]$. For the other direction, let $\mathcal{F} \in [S \cup T]$. Then $\mathcal{F}$ is an ultrafilter containing $S \cup T$. Since $\mathcal{F}$ is an ultrafilter, we have that it either contains S or S^c, T, or T^c. Notice that it cannot contain S^c and T^c, since if it did it would contain $S^c \cap T^c = (S \cup T)^c$, and so it would have the empty set, a contradiction. Hence, it must have either S or T, and so must either be in $[S]$ or $[T]$. Thus, $[S \cup T] \subset [S] \cup [T]$. We get $[S \cup T] = [S] \cup [T]$.
  (d) Notice that $S \cap T \subset S$, $S \cap T \subset T$, so we have $[S \cap T] \subset [S]$, $[S \cap T] \subset [T]$, and therefore $[S \cap T] \subset [S] \cap [T]$. For the other direction, let $\mathcal{F} \in [S] \cap [T]$. Then $\mathcal{F}$ is an ultrafilter which must contain S and T, and so by filter properties it must contain $S \cap T$. Hence, $\mathcal{F} \in [S \cap T]$, and so $[S] \cap [T] \subset [S \cap T]$. Thus, we get $[S \cap T] = [S] \cap [T]$.
  (e) Notice $[S^c] \cup [S] = [S^c \cup S] = [\mathbb{N}] = U\mathbb{N}$. Likewise, $[S^c] \cap [S] = [S^c \cap S] = [\varnothing] = \varnothing$. Thus, $[S^c] = [S]^c$.

(3) Notice that, by (2), we always have that

$$\bigcup [S_n] \subset \left[\bigcup S_n\right],$$

so it suffices to find a sequence such that the LHS is strictly smaller than the right. Let $S_n = \{n\}$, then we have that $[\bigcup S_n] = [\mathbb{N}] = U\mathbb{N}$ by (1). Take $\mathcal{F} \in U\mathbb{N}$; then we have that it is any ultrafilter on $\mathcal{N}$. In particular, take $\mathcal{F}$ to be an ultrafilter which is not principal (this exists since $\mathbb{N}$ is infinite, and any ultrafilter containing the cofinite filter cannot be principal). Then we have that $\mathcal{F} \notin \bigcup [S_n]$, and so the LHS is strictly smaller.

(4) Using the example from (3), we have

$$\left[\bigcap S_n^c\right] = \left[\left(\bigcup S_n\right)^c\right] = \left[\bigcup S_n\right]^c \neq \left(\bigcup [S_n]\right)^c = \bigcap [S_n]^c = \bigcap [S_n^c],$$

and so we are done.

□

Problem 47. Assume the notation of the prior problem.

(1) Show that $\{[S] \;:\; S \subset \mathbb{N}\}$ gives a base for a topology on $U\mathbb{N}$.
(2) Show that all the sets $[S]$ are both open and closed in $U\mathbb{N}$.
(3) Show that $U\mathbb{N}$ is compact.
(4) For $n \in \mathbb{N}$, let $\mathcal{F}_n = \{S \subset \mathbb{N} \;:\; n \in S\}$. Show that $\mathcal{F}_n$ is an ultrafilter on $\mathbb{N}$.
(5) Show that $\{\mathcal{F}_n \;:\; n \in \mathbb{N}\}$ is dense in $U\mathbb{N}$.
(6) Show that for every compact Hausdorff space K and every function $f : \mathbb{N} \to K$, there is a continuous function $\widetilde{f} : U\mathbb{N} \to K$ such that $\widetilde{f}(\mathcal{F}_n) = f(n)$ for every $n \in \mathbb{N}$. Deduce that $U\mathbb{N}$ is homeomorphic to the Stone-Cech compactification $\beta\mathbb{N}$.

Proof. (1) We need to show three things for this to be a base:

(a) First, from the prior problem it's clear that $\varnothing$ and $U\mathbb{N}$ is in $\mathcal{B}$.
(b) For each $a \in U\mathbb{N}$, there is a $B \in \mathcal{B} = \{[S] \;:\; S \subset \mathbb{N}\}$ such that $a \in B$. Since a is a filter, it must either be trivial or contain a non-empty set. If it is trivial, we see that $\varnothing \subset \mathbb{N}$ is such that $a \in [\varnothing] = \varnothing$. If it is non-trivial, we have that it contains a set S, and so we get $a \in [S] = B \in \mathcal{B}$.
(c) For any $B_0, B_1 \in \mathcal{B}$ and any $x \in B_0 \cap B_1$, there is some $B \in \mathcal{B}$ such that $x \in B \subset B_0 \cap B_1$. Take $[S_0] = B_0$, $[S_1] = B_1$. Then we have that $[S_0] \cap [S_1] = [S_0 \cap S_1]$. Assume it is not trivial (for if it is, the result is clear). Then we have some $a \in [S_0 \cap S_1] \subset [S_0] \cap [S_1]$, and $[S_0 \cap S_1] \in \mathcal{B}$, since $S_0 \cap S_1 \subset \mathbb{N}$ still.

Since the two properties are satisfied, this is does form a base for some topology.

(2) We have that $[S]$ is open in this topology. Furthermore, $[S^c]$ is also open, but by the prior problem $[S^c] = [S]^c$, so we get that $[S]$ is also closed.

(3) (Following the proof from the notes) Let $\bigcup A_\alpha$ be a cover of $U\mathbb{N}$. Suppose for contradiction that it admits no finite refinement. Let I be the indexing set for the α. Then we have that there is no n such that

$$[A_{\alpha_1} \cup \cdots \cup A_{\alpha_n}] = U\mathbb{N} = [\mathbb{N}].$$

Notice that, from the properties proven from the last problem, we have

$$[A_{\alpha_1} \cup \cdots \cup A_{\alpha_n}] = [A_{\alpha_1}] \cup \cdots \cup [A_{\alpha_n}].$$

So $A_{\alpha_1} \cup \cdots \cup A_{\alpha_n} \neq \mathbb{N}$ by prior properties. But then we get $A_{\alpha_1}^c \cap \cdots \cap A_{\alpha_n}^c \neq \varnothing$ for any finite n. By properties of filters, we can get a filter generated from $\{A_\alpha^c\}$, and thus an ultrafilter, say $\mathcal{F}$. Since $U\mathbb{N}$ is the space of all ultrafilters, we get that $\mathcal{F} \in U\mathbb{N}$. Now, we use the fact that $\bigcup A_\alpha = U\mathbb{N}$ to note that this filter must be in $[A_\alpha]$ for some α. But this implies that $A_\alpha, A_\alpha^c \in \mathcal{F}$, which is a contradiction. So there must be some finite refinement.

(4) We first show it's a filter. First, every set $S \in \mathcal{F}_n$ must be non-trivial, since $n \in S$. Next, take $A, B \in \mathcal{F}_n$. Then we have that $A \cap B \neq \varnothing$ and $A \cap B \in \mathcal{F}_n$, since $n \in A \cap B$. Finally, if $A \in \mathcal{F}_n$, B such that $A \subset B$, then $n \in B$, so $B \in \mathcal{F}_n$. Hence, it is a filter.

To see it's an ultrafilter, let $A \subset \mathbb{N}$. Then either $n \in A$ or $n \notin A$. If $n \in A$, then $A \in \mathcal{F}_n$ and we're done. If $n \notin A$, then $n \in A^c$, and so $A^c \in \mathbb{N}$. Since this applies for any set $A \subset \mathbb{N}$, we have that the equivalence established earlier gives that this is an ultrafilter.

(5) We wish to show that $\overline{\{\mathcal{F}_n \ : \ n \in \mathbb{N}\}} = U\mathbb{N}$. Let $\mathcal{F} \in U\mathbb{N}$. Then we have that, assuming $\mathcal{F}$ non-trivial, there is a set S such that $\mathcal{F} \subset [S]$. Since $\mathcal{F}$ non-trivial, we get that $S \in \mathcal{F}$, $S \neq \varnothing$. So taking $n \in S$, we get that $\mathcal{F}_n \in [S]$. Since the choice of $\mathcal{F} \in U\mathbb{N}$ was arbitrary, we get that $\{\mathcal{F}_n\}$ is dense.

(6) **Step 1:** We note that the image of an ultrafilter is an ultrafilter. Let $f : X \to Y$ be a function between sets. Let $\mathcal{G}$ be a filter on Y, $\mathcal{F}$ an ultrafilter on X, then we want to show that if $f(\mathcal{F}) \subset \mathcal{G}$, we have $f(\mathcal{F}) = \mathcal{G}$. This, however, follows by the fact that the set $U \in G$, $U \notin f(\mathcal{F})$ implies $f^{-1}(U) \notin \mathcal{F}$, so $f^{-1}(U)^c = f^{-1}(U^c) \in \mathcal{F}$, which means that $U^c \in f(\mathcal{F})$, but this means that $U, U^c \in \mathcal{G}$, which is a contradiction. Hence, there are no $U \in G$ which are not in $\mathcal{F}$, and so $f^{-1}(\mathcal{F}) = \mathcal{G}$.

Step 2: We show that if a space K is compact, then every ultrafilter is convergent. We follow the proof given in the notes linked on the homework. That is, let $\mathcal{F}$ be an ultrafilter on a compact space X without a limit point. Then each $x \in X$ has a neighborhood U_x containing no element of $\mathcal{F}$. Using compactness, we can construct a finite refinement of the U_x covering X to get $\{U_{x_i}\}_{i=1}^n$. Fix a set $A \in \mathcal{F}$. Then $A \subset \bigcup_{i=1}^n U_{x_i}$. Using the quiz problem, we have that

$$\bigcup_{i=1}^{n} (A \cap U_{x_i}) = A \in F \implies A \cap U_{x_i} \in \mathcal{F} \text{ for some i,}$$

which is a contradiction. So there is some point where it converges to.

Remark. Recall that a filter on a Hausdorff space can converge to only at most one point. So every ultrafilter converges to *exactly* one point. Furthermore, the Hausdorff property proves that there is only one such extension function.

Step 3: We again follow the proof given in the notes linked in the homework. Let $\mathcal{F} \in U\mathbb{N}$. By **Step 1**, we have that $f(\mathcal{F})$ is an ultrafilter on K. Since $f(\mathcal{F})$ is an ultrafilter on a compact space, **Step 2** tells us that it is convergent to some $x \in K$. Hence, define $\widetilde{f}(\mathcal{F}) = \lim f(\mathcal{F})$. Notice that defining it this way gives us that

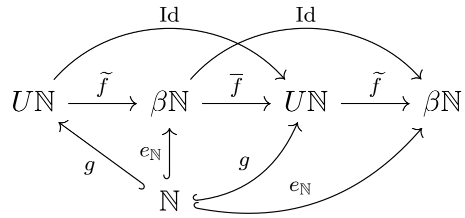
$$\widetilde{f}(\mathcal{F}_n) = \{f(S) \ : \ S \subset \mathbb{N}, \ n \in S\}, \ f(n) \in \{f(S) \ : \ S \subset \mathbb{N}, \ n \in S\}$$
$$\widetilde{f}(F_n) = \lim(\mathcal{F}_n) = f(n).$$

So this is a well-defined function, and is indeed an extension of f.

Step 4: We need to show that $\widetilde{f}$ is continuous. Again, we show it's continuous the same way as the notes linked in the homework set. We show it's continuous in the following way: Let $\mathcal{F} \in U\mathbb{N}$ be a filter. Since $\widetilde{f}(\mathcal{F}) = \lim f(\mathcal{F}) \in K$, let U be an open neighborhood of $\widetilde{f}(\mathcal{F})$. Since K compact, pick $V \subset U$ such that $\overline{V} \subset U$. Since $f(\mathcal{F})$ converges, we have that there is a set $A \in \mathcal{F}$ such that $f(A) \subset V \subset U$. Notice as well that $A \in \mathcal{F}$ implies that $\mathcal{F} \in [A]$, and so $[A]$ acts as an open neighborhood of $\mathcal{F}$. We'll show that all the points in $[A]$ are mapped into U. Let $\mathcal{G} \in [A]$. Then $A \in \mathcal{G}$, and furthermore $f(A) \in f(\mathcal{G})$. We have that $\widetilde{f}(\mathcal{G}) = \lim f(\mathcal{G}) \in \overline{f(A)} \subset \overline{V} \subset U$. Hence, $\widetilde{f}$ is continuous.

Step 5: We need to show that $U\mathbb{N}$ is Hausdorff. Let $\mathcal{F}, \mathcal{G}$ be two filters such that $\mathcal{F} \neq \mathcal{G}$. Then there is a set $A \in \mathcal{F}$, such that $A \notin \mathcal{G}$. Since $\mathcal{G}$ is an ultrafilter, we must have that $A^c \in \mathcal{G}$. Hence, $\mathcal{F} \in [A]$, $G \in [A]^c$, $[A] \cap [A^c] = \varnothing$, so the space is Hausdorff.

We deduce that $U\mathbb{N}$ is homeomorphic to $\beta\mathbb{N}$ in the following way: let $g : \mathbb{N} \to U\mathbb{N}$ via $g(n) = \mathcal{F}_n$. This mapping is well-defined and injective. We can then invoke the universal property using existence and uniqness to get the following (messy) commutative diagram:



Hence, we have that $\widetilde{f} \circ \overline{f} = \text{Id}$, $\overline{f} \circ \widetilde{f} = \text{Id}$, and these are continuous, so we have that they are homeomorphisms. Hence, the spaces are homeomorphic.

□

Problem 48. Suppose X is a normed space and $Y \subset X$ is a subspace. Define $Q : X \to X/Y$ by $Q(x) = x + Y$. Define

$$||Q(x)||_{X/Y} = \inf\{||x - y||_X \ : \ y \in Y\}.$$

(1) Prove that $|| \cdot ||_{X/Y}$ is a well-defined seminorm.
(2) Show that if Y is closed, then $|| \cdot ||_{X/Y}$ is a norm.
(3) Show that in the case of (2) above, $Q : X \to X/Y$ is continuous and open.
(4) Show that if X is Banach, so is X/Y.

Proof. We first remark that this is sending a point to it's equivalence class in the quotient space.

(1) It's clear that it's well-defined, since regardless of representation in the class, we get the same result. Next, since $|| \cdot ||_X \geq 0$ for all $x \in X$, we have that $|| \cdot ||_{X/Y} \geq 0$. Next, letting $c \in F - \{0\}$, we note that $cQ(x) = c(x + Y) = cx + cY = cx + Y$, so

$$||cQ(x)||_{X/Y} = \inf\{||cx - y||_X \ : \ y \in Y\} = \inf\{||c(x - c^{-1}y)||_X \ : \ y \in Y\}$$

$$= \inf\{|c|||x - c^{-1}y||_X \ : \ y \in Y\} = |c| \inf\{||x - y||_X \ : \ y \in Y\} = |c|||Q(x)||_{X/Y}.$$

In the case $c = 0$, we have $0Q(x) = 0(x+Y) = 0$, and $||0||_{X/Y} = 0$, which matches. Finally, let $x, z \in X$, then $Q(x + z) = x + z + Y = (x + Y) + (z + Y) = Q(x) + Q(z)$, so

$$||Q(x + z)||_{X/Y} = \inf\{||(x + z) - y||_X \ : \ y \in Y\} = \inf\{||(x - y) + (z - w)||_X \ : \ y, w \in Y\}$$

$$\leq \inf\{||(x - y)||_X + ||(z - w)||_X \ : \ y, w \in Y\} = \inf\{||x - y||_X \ : \ y \in Y\} + \inf\{||z - y||_X \ : \ y \in Y\}$$

$$= ||Q(x)||_{X/Y} + ||Q(z)||_{X/Y}.$$

Hence, we have that this is a seminorm.

(2) Let $Q(x)$ be such that $||Q(x)||_{X/Y} = 0$. Then this means that

$$\inf\{||x - y||_X \ : \ y \in Y\} = 0.$$

Since this is an infimum, this means we can construct a sequence $\{y_n\}$ such that $||x-y_n||_X \to 0$. But since $|| \cdot ||_X$ is a norm, this means that $y_n \to x$. Since Y is closed, this means that $x \in Y$. Hence, $||Q(x)||_{X/Y} = 0$ if and only if $Q(x) = 0$, or $x \in Y$.

(3)

Remark. The second part is based on the following Stackexchange post:
https://math.stackexchange.com/questions/3128639/is-projection-to-quotient-space-an-open-map

We say in (1) that the projection map was linear. To show continuity, it suffices to show that it's bounded. That is, we have $||Q(x)||_{X/Y} \leq c||x||_X$ for some $c \geq 0$. Notice that

$$||Q(x)||_{X/Y} = \inf\{||x - y||_X \ : \ y \in Y\} \leq ||x||_X + \inf\{||y||_X \ : \ y \in Y\} = ||x||_X.$$

Hence, Q is bounded, and so continuous.

Next, we need to show that it's an open map. Let $U \subset X$ open. Then we need to show that $Q(U)$ is open. Let $x + Y = Q(x) \in Q(U)$, then $x \in U$. Let $\epsilon > 0$ be such that $B_\epsilon(x) \subset U$. Let $z + Y \in B_\epsilon(x + Y)$. Then we have that

$$||Q(x) - Q(z)||_{X/Y} = ||Q(x - z)||_{X/Y} < \epsilon,$$

so there is a $y \in Y$ so that $||x - z - y||_X < \epsilon$. Hence,

$$||Q(x - z)||_{X/Y} \leq ||x - z - y||_X < \epsilon,$$

so we get $z - y \in B_\epsilon(x)$, but this tells us that $Q(z) \in Q(B_\epsilon(x)) \subset Q(U)$. Since this applies for all such z in $B_\epsilon(Q(x))$, we have $B_\epsilon(Q(x)) \subset Q(U)$, and since this applies for all $Q(x) \in Q(U)$, we get that $Q(U)$ is open.

(4) (Presumably here we also take Y to be closed still.) We have that X, X/Y are both normed vector spaces. In the case that X is Banach, we have that it's complete. Hence, it suffices to show that X/Y is complete as well. **Theorem 5.1** states that a normed vector space X/Y is complete if and only if every absolutely convergent series in X/Y converges. Take an absolutely convergent series $\{x_n + Y\}$ in X/Y. That is, we have

$$\sum ||x_n + Y||_{X/Y} = \sum ||Q(x_n)||_{X/Y} < \infty.$$

Choose representatives $x_n \in x_n + Y$ such that $||x_n||_X \leq ||Q(x_n)||_{X/Y} + 2^{-n}$. Then we get

$$\sum ||x_n||_X \leq \sum ||Q(x_n)||_{X/Y} + 2^{-n} < \infty.$$

Since X is complete, we use this to note that $\sum x_n$ converges to some $x \in X$. Next, notice that

$$\sum_1^N Q(x_n) = Q\left(\sum_1^N x_n\right)$$

by linearity, and so using the fact that Q is continuous we get

$$\lim_{N\to\infty} \sum_1^N Q(x_n) = \lim_{N\to\infty} Q\left(\sum_1^N x_n\right) = Q\left(\sum x_n\right) = Q(x).$$

So, every absolutely convergent series in X/Y converges, and hence X/Y is complete.

□

Problem 49. Suppose F is a finite dimensional vector space.

(1) Show that for any two norms $||\cdot||_1$, $||\cdot||_2$ on F, there is a $c > 0$ such that $||f||_1 \leq c||f||_2$ for all $f \in F$. Deduce that all norms on F induce the same vector space topology on F.
(2) Show that for any two finite dimensional normed space F_1, F_2, all linear maps $T : F_1 \to F_2$ are continuous.
(3) Let X, F be normed space with F finite dimensional, and let $T : X \to F$ be a linear map. Prove that the following are equivalent:
   (a) T is bounded (there is an $R > 0$ such that $T(B_1(0_X)) \subset B_R(0_F))$), and
   (b) $\ker(T)$ is closed.

Proof. (1) We follow **Exercise 6** in Folland to show the equivalence of norms. Let $\mathrm{Dim}(F) = n$, then we have a basis $\{e_1, \ldots, e_n\}$. Define $||\cdot||_1$ via

$$\left|\left|\sum_1^n a_i e_i\right|\right|_1 = \sum_1^n |a_i|.$$

We first show that this is a norm. We need to show four properties. Let $v, w \in F$ be

$$v = \sum_1^n a_i e_i,$$

$$w = \sum_1^n b_i e_i.$$

(a) We have $||v||_1 \geq 0$, since $|\cdot| \geq 0$.
(b) For c in the field, we have

$$||cv||_1 = \sum_1^n |ca_i| = |c| \sum_1^n |a_i| = |c| ||v||_1.$$

(c) We have

$$||v + w||_1 = \sum_1^n |a_i + b_i| \leq \sum_1^n |a_i| + |b_i| = ||v||_1 + ||w||_1.$$

(d) If $||v||_1 = 0$, then this means that $|a_i| = 0$ for all i, which only happens if $a_i = 0$, and so $v = 0$.

So $|| \cdot ||_1$ is a norm.

Notice that it suffices to show this for $|| \cdot ||_1$ and another norm $|| \cdot ||$. Notice we have as well that

$$f = \sum_1^n a_i e_i.$$

Hence,

$$||f|| = \left|\left| \sum_1^n a_i e_i \right|\right| = \sum_1^n |a_i| ||e_i|| \leq \max\{||e_i||\} \sum_1^n |a_i| = \max\{||e_i||\} ||f||_1.$$

Letting $c = \max\{||e_i||\}$ gives us the desired result.

Let $K = \{x \in \mathbb{C}^n \ : \ ||x||_2 = 1\}$, where $|| \cdot ||_2$ is the standard Euclidean norm. This is compact by assumption. Let $T : \mathbb{C}^n \to F$ be a isomorphism. Notice that this is continuous with respect to the norm $|| \cdot ||$, since take $z \in K$ we have

$$||T(z)|| \leq \left|\left| \sum a_i T(e_i) \right|\right| \leq |a| \sum ||T(e_i)|| \leq \sum ||T(e_i)||,$$

and so it's bounded since this is finite.

Next, since K is compact, we have that the infimum is realized. Hence, there is some $z \in K$ such that $T(z) = \epsilon$, and for all other $z \in K$ we get $T(z) \geq \epsilon$. Now, take $f \in F$ arbitrary. Since T is an isomorphism, there is a z such that $T(z) = f$. Hence,

$$||f|| = ||T(z)|| = ||T(z/||z||_2)|| \cdot ||z||_2 \geq \epsilon ||z||.$$

Notice that we then get

$$||f|| \geq \epsilon ||z|| = \epsilon \sqrt{\sum |a_i|^2},$$

where the a_i are the coefficients, and we have from the arithmetic-geometric inequality

$$\epsilon \sqrt{\sum |a_i|^2} \geq \frac{\epsilon}{n} \sum |a_i|^2 = \frac{\epsilon}{n} ||f||_1.$$

So every norm is equivalent to $||f||_1$. Every norm being equivalent to $||f||_1$ is sufficient, since for two norms $||\cdot||_1$, $||\cdot||_2$,

$$c||f|| \leq ||f||_1 \leq C||f||,$$

$$c'||f|| \leq ||f||_2 \leq C'||f||,$$

and so

$$\frac{c'}{C}||f||_1 \leq ||f||_2 \leq \frac{C'}{c}||f||_1,$$

hence, they are equivalent. This also gives us the conditions to, for every norm, find a $c > 0$ such that $||f||_1 \leq c||f||_2$. By Homework 1, the topologies are then equivalent.

(2) Let $(F_1, ||\cdot||_1)$, $(F_2, ||\cdot||_2)$ be two normed spaces. Consider the map $T : F_1 \to F_2$. We wish to show they are continuous. Letting $\{e_i\}$ be a basis for F_1, we have

$$||T(x)||_2 = \left|\left|T\left(\sum a_i e_i\right)\right|\right|_2 = \left|\left|\sum a_i T(e_i)\right|\right|_2 \leq \sum |a_i|||T(e_i)||_2.$$

Taking $M = \max\{T(e_i)\}$, we have

$$\sum |a_i|||T(e_i)||_2 \leq M \sum |a_i| \leq M||x||,$$

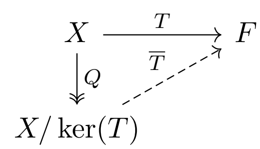
where $||x||$ is the norm given in (1) on F_1. Since every norm is equivalent to this one by (1), we have

$$M||x|| \leq CM||x||_1,$$

where $C > 0$ is some constant. Hence, T is bounded, and so is continuous.

(3) $(a) \implies (b)$: T bounded implies continuous, $\{0\} \subset F$ is closed, so we have $T^{-1}(\{0\}) = \ker(T)$ is closed.

$(b) \implies (a)$: Notice that the isomorphism theorem gives us that



Since $\ker(T)$ is closed, we have that **Problem 3** gives us that Q is continuous (and open). Since T is linear, we have that $T(X) \leq F$ is a subspace (so finite dimensional), and so $X/\ker(T) \cong T(X)$ is a finite dimensional vector space. By (2) of this problem, this gives us that $\overline{T}$ is continuous. Since this diagram commutes, we get $T = \overline{T} \circ Q$, and since Q and $\overline{T}$ are both continuous, T must be continuous. By the equivalence, this tells us that T is bounded.

□

Problem 50. Suppose X is a Banach space and $T \in \mathcal{L}(X) = \mathcal{L}(X, X)$. Let $I \in \mathcal{L}(X)$ be the identity map.

(1) Show that if $||I - T|| < 1$, then T is invertible.
(2) Show that if $T \in \mathcal{L}(X)$ is invertible and $||S - T|| < ||T^{-1}||^{-1}$, then S is invertible.
(3) Deduce that the set of invertible operators $GL(X) \subset \mathcal{L}(X)$ is open.

Proof. (1) We proceed via the hint. Notice that $||I - T|| < 1$ implies that the series

$$\sum ||I - T||^n < \infty.$$

Since X is a Banach space, we have that it's dual $\mathcal{L}(X)$ is also a Banach space. Hence, we have that, since this is an absolutely convergent series, the series

$$\sum_0^N (I-T)^n \to S \in \mathcal{L}(X).$$

We now notice that

$$\begin{aligned} ||ST - I|| &= ||ST - S + S - I|| = ||S(T-I) + S - I|| \\ &= \lim_{N\to\infty} \left|\left| \left(\sum_0^N (I-T)^n\right)(T-I) + \sum_0^N (I-T)^N - I \right|\right|. \end{aligned}$$

Notice that

$$\sum_0^N (I-T)^N (T-I) = -\sum_1^{N+1} (I-T)^N,$$

and hence we're left with

$$\begin{aligned} \lim_{N\to\infty} \left|\left| -\sum_1^{N+1} (I-T)^n + \sum_0^N (I-T)^N - I \right|\right| &= \lim_{N\to\infty} \left|\left|(I-T)^0 - (I-T)^{N+1} - I\right|\right| \\ &= \lim_{N\to\infty} ||(I-T)||^{N+1} = 0. \end{aligned}$$

Hence, we have that $ST = I$. Analogously, we need to show that $TS = I$. We see

$$\begin{aligned} ||TS - I|| &= ||TS - S + S - I|| = ||(T-I)S + S - I|| \\ &= \lim_{N\to\infty} \left|\left| (T-I)\left(\sum_0^N (I-T)^n\right) + \sum_0^N (I-T)^n - I \right|\right| \\ &= \lim_{N\to\infty} \left|\left| -\left(\sum_1^{N+1} (I-T)^n\right) + \sum_0^N (I-T)^n - I \right|\right| \\ &= \lim_{N\to\infty} \left|\left|I - (I-T)^{N+1} - I\right|\right| \lim_{N\to\infty} ||(I-T)||^{N+1} = 0. \end{aligned}$$

Hence, $TS = I$, so T is invertible.

(2) Since T is invertible, we have that T^{-1} is well-defined. Then we get

$$||T^{-1}S - I|| = ||T^{-1}S - T^{-1}T|| \le ||T^{-1}|| \cdot ||S - T|| < ||T^{-1}|| \cdot ||T^{-1}||^{-1} = 1.$$

Hence, $T^{-1}S$ is invertible with inverse A, and so we have

$$A(T^{-1}S) = (AT^{-1})S = I,$$

so S is has left inverse AT^{-1}. Likewise,

$$S(AT^{-1}) = TT^{-1}SAT^{-1} = T(T^{-1}S)AT^{-1} = TT^{-1} = I,$$

so S has right inverse AT^{-1}, and so AT^{-1} is an inverse for S. Hence, S is invertible.

(3) To show that $GL(X)$ is open, we need to show that for every $T \in GL(X)$, there is an open ball U such that $U \subset GL(X)$. By (2), we showed that the open ball of radius $||T^{-1}||^{-1}$ centered at T is contained in $GL(X)$. Hence, we have that $GL(X)$ is open.

□

Remark. Thomas O'Hare was a collaborator.

Problem 51. Provide examples of the following:

(1) Normed vector spaces X and Y and a discontinuous linear map $T : X \to Y$ with closed graph.
(2) Normed vector spaces X and Y and a family of linear operators $\{T_\lambda\}_{\lambda\in\Lambda}$ such that $(T_\lambda(x))_{\lambda\in\Lambda}$ is bounded for every $x \in X$ but $(||T_\lambda||)_{\lambda\in\Lambda}$ is not bounded.

Proof. (1) We follow Folland 5.29. Let $Y = \{f : \mathbb{N} \to \mathbb{R} : \sum |f(n)| < \infty\}$, $X = \{f \in Y : \sum n|f(n)| < \infty\}$, both equipped with the norm $||\cdot||_1$, given by

$$||f||_1 = \sum |f(n)|.$$

Define $T : X \to Y$ by $T(f(n)) = nf(n)$. We wish to show that T is closed but not bounded, and hence not continuous. To show that T is closed, we need to show that $\Gamma(T) \subset X \times Y$ is closed. Take a sequence $((x_n, T(x_n))) \subset \Gamma(T)$, and suppose $(x_n, T(x_n)) \to y$. We wish to show that $y \in \Gamma(T)$. Notice that y is of the form (x, z), where $x_n \to x$, $T(x_n) \to z$, both converging in L^1 norm. We have that for all $\epsilon > 0$, there exists an N such that for all $n \geq N$,

$$|x(k) - x_n(k)| \leq ||x - x_n||_1 = \sum |x(k) - x_n(k)| < \epsilon.$$

Hence, we have that $x_n(k) \to x(k)$ pointwise. Similarly, we have $T(x_n) \to z$ in the L^1 norm, so for all $\epsilon > 0$ there exists an N such that for all $n \geq N$,

$$|T(x_n(k)) - z(k)| \leq ||T(x_n) - z||_1 = \sum |T(x_n(k)) - z(k)| < \epsilon,$$

and so we have that $T(x_n(k)) \to z(k)$ pointwise. Recall that we have $T(x_n(k)) = kx_n(k)$, and so taking the limit as $n \to \infty$ gives $T(x_n(k)) = kx_n(k) \to kx(k) = T(x(k)) = z(k)$, so $(x, z) \in \Gamma(T)$. Hence, it is closed.

Next, we wish to show that it's unbounded (unbounded here will be equivalent to discontinuous). Recall that being bounded means that, for all $x \in X$, there exists a $C > 0$ such that

$$||T(x)||_1 \leq C||x||_1,$$

but we have that

$$||T(x)||_1 = \sum |T(x(k))| = \sum |kx(k)| = \sum k|x(k)|.$$

So take, for example, the functions $\delta_n(x) = \delta_{nx}$, which is 1 if $n = x$ and 0 otherwise. We have $\delta_{nx} \in X$, since

$$\sum n|\delta_k(n)| = n < \infty,$$

but we see that

$$||T(\delta_k(n)||_1 = \sum_n |T(\delta_k(n)| = \sum_n |n\delta_k(n)| = \sum_n n|\delta_k(n)| = n,$$

while

$$||\delta_k(n)||_1 = \sum_n |\delta_k(n)| = 1.$$

So trying to choose such a C is impossible, since we can just take δ_{C+1} to contradict it. Hence, T is unbounded, and by the equivalence this means that T is not continuous.

(2) (Royden, Fitzpatrick, Exercise 41, Section 13.5) Let X be the space of all polynomials defined on $\mathbb{R}$. Let $||\cdot||$ be a norm on X defined by

$$||a_0 + a_1 x + \cdots + a_n x^n|| = \sum |a_i|.$$

To show it's a norm, we need to establish the following:

(a) Notice that if we let

$$p = a_0 + \cdots + a_n x^n,$$
$$q = b_0 + \cdots + b_m x^m,$$

and without loss of generality take $m \geq n$. We have then

$$p + q = (a_0 + b_0) + \cdots + (a_n + b_n)x^n + b_{n+1}x^{n+1} + \cdots + b_m x^m.$$

Hence,

$$||p+q|| = |(a_0 + a_1 + \cdots + a_n) + (b_0 + \cdots + b_m)| \leq |a_0 + \cdots + a_n| + |b_0 + \cdots + b_m| = ||p|| + ||q||.$$

(b) Let $r \in \mathbb{R}$. Then we have

$$||rp|| = \sum |ra_i| = \sum |r||a_i| = |r| \sum |a_i| = |r| \cdot ||p||.$$

(c) Finally, let p be such that $||p|| = 0$. Then this means that

$$\sum |a_i| = 0,$$

but this can only happen if $a_i = 0$ for all i. Hence, $p = 0$.

Thus, it's a norm.

For each n, define $T_n : X \to \mathbb{R}$ via $T_n(p) = \frac{\partial^n p}{\partial x^n}\big|_{x=0} = p^{(n)}(0)$. This is linear, since derivatives are linear. We then want to show that for all $p \in X$, we have that the sequence $(T_n(p))$ is bounded. Notice that

$$|T_n(p)| \leq \deg(p)! \cdot \max\{a_i \ : \ a_i \text{ is a coefficient of p}\}$$

for all n, and so it is bounded for every $p \in X$. We also have that

$$||T_n|| = \sup\{||T_n(p)|| \ : \ ||p|| = 1\} \geq n!,$$

since we can just take the polynomial $p = x^n$, and so $(||T_n||)$ is unbounded.

Remark. Thomas had a clever way of doing this using the same set up but the infinity norm, which I thought was much cleaner. The result is still the same.

□

Problem 52. Suppose X and Y are Banach space and $T : X \to Y$ is a continuous linear map. Show that the following are equivalent:

(1) There exists a constant $c > 0$ such that $||T(x)||_Y \geq c||x||_X$ for all $x \in X$.
(2) T is injective and has closed range.

Proof. (1) $\implies$ (2): We wish to show that T is injective and has closed range. Recall that a linear map is injective if its kernel is trivial. Thus, we examine $\ker(T) = \{x \in X \ : \ T(x) = 0\}$. Let $x \in X$, and assume that $T(x) = 0$. By assumption, we have

$$0 = ||T(x)||_Y \geq c||x||_X \geq 0,$$

so by norm properties this forces $x = 0$. Hence, $\ker(T) = 0$, so T injective.

Next, we wish to show that it has closed range. Let $(T(x_n)) \subset Y$ be a sequence such that $\lim T(x_n) = y$, then we wish to show that $y \in \operatorname{Im}(T)$. Notice that

$$||T(x_n) - T(x_m)||_Y \geq c||x_n - x_m||_X,$$

so we have that $(x_n) \subset X$ is a Cauchy sequence as well. Hence, we have that $x_n \to x$ by completeness, and so by continuity we have that $\lim T(x_n) \to T(x) = y$, so $y \in \mathrm{Im}(T)$. Thus, $\mathrm{Im}(T)$ is closed.
(2) $\implies$ (1) : Let $R = \mathrm{Im}(T) \subset Y$. This is a closed subset by assumption, and so Banach. We have that $T : X \to R$ is an isomorphism, and applying the open mapping theorem gives that it is a homeomorphism. Hence, $||T^{-1}|| = C < \infty$, and we have that $||T^{-1}(y)|| \le ||T^{-1}|| \cdot ||y|| = C||y||$. Since T is invertible, we have that for all $x \in X$ there is a y so that $T^{-1}(y) = x$ and $T(x) = y$, and so we have that $||x|| \le C||T(x)||$ for all $x \in X$. Thus, we get $\frac{1}{C}||x|| \le ||T(x)||$ for all $x \in X$. □

Problem 53. Suppose $\varphi, \varphi_1, \ldots, \varphi_n$ are linear functionals on a vector space X. Prove that the following are equivalent:

(1) $\varphi = \sum_{k=1}^n a_k \varphi_k$, where $a_i \in F$, F the underlying field (i.e. $\varphi \in \mathrm{span}\{\varphi_1, \ldots, \varphi_n\}$).
(2) There is an $a > 0$ such that for all $x \in X$, $|\varphi(x)| \le a \max_{k=1,\ldots,n} |\varphi_k(x)|$.
(3) $\bigcap_{k=1}^n \ker(\varphi_k) \subset \ker(\varphi)$.

Proof. (1) $\implies$ (2): Take $x \in X$, then we have

$$\varphi(x) = \sum_{k=1}^n a_k \varphi_k(x).$$

Let $a = n \cdot \max_{k=1,\ldots,n} |a_k|$. Then we have for all $x \in X$,

$$|\varphi(x)| = \left| \sum_{k=1}^n a_k \varphi_k(x) \right| \le \sum_{k=1}^n |a_k||\varphi_k(x)| \le \max_{k=1,\ldots,n} |\varphi_k(x)| \sum_{k=1}^n |a_k| \le a \max_{k=1,\ldots,n} |\varphi_k(x)|.$$

(2) $\implies$ (3) : Let x be such that $\varphi_k(x) = 0$ for $k = 1, \ldots, n$. Then we have

$$0 \le |\varphi(x)| \le a \cdot 0 = 0 \implies \varphi(x) = 0,$$

hence $x \in \ker(\varphi)$. So we have $\bigcap_{k=1}^n \ker(\varphi_k) \subset \ker(\varphi)$.
(3) $\implies$ (1) (Royden, Fitzpatrick, Proposition 4, Section 14.1): We go by induction. We first show it for the base case, $n = 1$. Assume that $\varphi \ne 0$, since the result holds clearly if $\varphi = 0$. Choose $x_0 \ne 0$ for which $\varphi(x_0) = 1$. Then by assumption $\varphi_1(x_0) \ne 0$. Notice as well that $\varphi_1 : X \to F$, so $X = \ker(\varphi_1) \oplus \mathrm{span}(x_0)$. Defining $a_1 = 1/\varphi_1(x_0)$, we get

$$\varphi(x) = \varphi(y + tx_0) = a_1\varphi_1(y + tx_0) = a_1\varphi_1(x),$$

so $\varphi = a_1\varphi_1$. Hence, it holds for the base case.

Now, assume for the induction hypothesis that it holds up to $n-1$. Then assume without loss of generality that $\varphi_n \ne 0$, since the result holds clearly in this case by the induction hypothesis. Choose $x_0 \in X$ such that $\varphi_n(x_0) = 1$. Then, using again the fact that $\varphi_n : X \to F$ is linear, we get that $X = \ker(\varphi_n) \oplus \mathrm{span}(x_0)$, so

$$\bigcap_{i=1}^{n-1} (\ker(\varphi_i) \cap \ker(\varphi_n)) \subset \ker(\varphi) \cap \ker(\varphi_{n-1}).$$

By the induction hypothesis,

$$\varphi = \sum_{i=1}^{n-1} a_i \varphi_i$$

on $\ker(\varphi_n)$, $a_i \in F$, so letting $a_n = \varphi(x_0) - \sum_{i=1}^{n-1} a_i\varphi_i(x_0)$, we get

$$\varphi = \sum_{i=1}^{n} a_i \varphi_i$$

on all of X, using a similar substitution to above. □

Problem 54. Let X be a normed space.

(1) Show that every weakly convergent sequence in X is norm bounded.
(2) Show that every weak* convergent sequence in X^* is norm bounded.

Proof. (1) (Royden, Fitzpatrick, Theorem 12 Section 14.2) Let (x_n) be a weakly convergent sequence in X, and suppose it converges to x. Then this means that, for all $\varphi \in X^*$, we have that $\varphi(x_n) \to \varphi(x)$. Per Royden, we let $J : X \to (X^*)^*$ be the map which sends a point to the evaluation map; that is, $J(x) : X^* \to F$ such that $J(x)(\varphi) = \varphi(x)$. Notice that for fixed $\phi \in X^*$, we get that $J(x_n)(\phi) = \phi(x_n) \to \phi(x) = J(x)(\phi)$, so it pointwise converges to $J(x)$, so $||J(x_n)(\varphi)|| < \infty$ for all φ. By the Banach-Steinhaus/Uniform Boundedness Principle, we notice that $||J(x_n)|| < \infty$. Since J is an isometry (by corollaries of Hahn-Banach) we have that $||x_n|| < \infty$.

Remark. I believe that for the next part we need to assume X is Banach, since these notes provide a counterexample to the case where X not Banach.
https://people.math.gatech.edu/∼heil/handouts/weak.pdf

(2) Recall that weak* convergence means that a sequence $(\varphi_n) \subset X^*$ converges to $\varphi \in X^*$ if, for all $x \in X$, we have $\varphi_n(x) = \varphi(x)$. We wish to show that $||\varphi_n|| < \infty$. Since $\varphi_n(x) = \varphi(x)$, $||\varphi(x)|| < \infty$ by assumption, we have that $\sup_n ||\varphi_n(x)|| < \infty$, and since this applies for all $x \in X$, we get that the Uniform Boundedness Principle implies that $||\varphi_n|| < \infty$. Hence, the sequence is bounded.

□

Problem 55. Let X be a normed vector space with closed unit ball B. Let B^{**} be the unit ball in X^{**}, and let $i : X \to X^{**}$ be the canonical inclusion. Show that $i(B)$ is weak* dense in B^{**}.

Proof. (Royden, Fitzpatrick, Theorem 6, Section 15.3) We have that

$$B^{**} = \{\hat{x} \ : \ ||\hat{x}|| \leq 1\}.$$

We first want to establish that B^{**} is weak* closed.

Remark. The following is what I wanted to initially do, which I think still works. However, Thomas brought up that this should just be a direct result from Banach-Alaoglu.

Take a net $(\hat{x}_n) \subset B^{**}$ which converges weak* to $\hat{x}$. We wish to show that $\hat{x} \in B^{**}$. Since $(\hat{x}_n)$ converges weak* to $\hat{x}$, we get that this means that $\lim \hat{x}_n(\varphi) = \hat{x}(\varphi) = \varphi(x)$. Notice that the continuity of norms gives us $||\hat{x}(\varphi)|| = \lim ||\hat{x}_n(\varphi)|| = \lim ||\varphi||||\hat{x}_n|| \leq \lim ||\varphi|| = ||\varphi||$. Thus, we see that

$$||\hat{x}|| := \sup\{||\hat{x}(\varphi)|| \ : \ ||\varphi|| = 1\} \leq 1,$$

so $\hat{x} \in B^{**}$. Hence, it is closed.

Recall that we have i is an isometry (corollaries of Hahn-Banach), so $i(B) \subset B^{**}$. So taking the weak* closure of $i(B)$ and denoting it C, we get that $C \subset B^{**}$, since B^{**} is closed by above. Notice that $B = \{x \in X \ : \ ||x|| \leq 1\}$ is convex, and the linear image of a convex set is convex, since taking $a, b \in i(B)$, we have $x, y \in X$ such that $i(x) = a$, $i(y) = b$, so we see that $ta + (1-t)b \in i(B)$ for all $t \in [0, 1]$, since $ti(x) + (1-t)i(y) = i(tx + (t-y)y) \in i(B)$, using the fact that B is convex. Notice as well that the closure of a convex set is convex; letting X be our convex set, we take $(x_n), (y_n) \subset X$ such that $x_n \to x \in \overline{X}$, $y_n \to y \in \overline{X}$. Then we see that $tx + (1-t)y \in \overline{X}$ for all $t \in [0, 1]$, since $tx_n + (1-t)y_n \in X$ for all n, $t \in [0, 1]$, and so taking limits gives us what we want. Thus, C is a convex closed set with respect to the weak* topology.

Remark. As Thomas pointed out, there seems to be a typo in what Royden actually does. This fix comes from
http://mathonline.wikidot.com/goldstine-s-theorem
which, as far as I can tell, still uses the same separation theorem.

Suppose now for contradiction that $C \neq B^{**}$; that is, $i(B)$ is not weak* dense in B^{**}. Let $\varphi \in B^{**} - C$. Using the so called "Hyperplane Separation theorem," (page 292 of Royden, Fitzpatrick, Corollary 26 specifically) we have that there is some linear functional T on C such that $||T|| = 1$ and $T(\varphi) < \inf_{\alpha \in C} T(\alpha)$. Since $i(B) \subset C$, we have $\inf_{\alpha \in C} T(\alpha) \leq \inf_{\alpha \in i(B)} T(\alpha) = -1$. Hence, we get that $T(\varphi) < -1$, and so $||T(\varphi)|| > 1$. We chose T such that $||T|| = 1$, and ϕ was chosen such that $||\phi|| \leq 1$, so we have that $||T(\varphi)|| \leq ||T|| \cdot ||\varphi|| \leq 1$. This gives us a contradiction. Hence, there cannot exist such a φ, and so $B^{**} = C$. $\square$

Remark. Thomas O'Hare was a collaborator.

Problem 56. Show that there is a $\varphi \in (l^\infty)^* = \mathcal{L}(l^\infty, \mathbb{R})$ satisfying the following two conditions:

(1) Letting $S : l^\infty \to l^\infty$ be the shift operator $(Sx)_n = x_{n+1}$ for $x = (x_n)_{n\in\mathbb{N}}$, $\varphi = \varphi \circ S$.
(2) For all $x \in l^\infty$, $\liminf x_n \leq \varphi(x) \leq \limsup x_n$.

Proof.

Remark. I used a modified version of these notes:
http://homepages.math.uic.edu/~furman/4students/Banach-LIM.pdf

We try using the Hahn-Banach theorem. Let

$$p((x_n)) = \limsup_{n\to\infty} \left| \frac{1}{n} \sum_{i=1}^n x_i \right|.$$

We need to show that this gives us a sublinear functional. Notice first that $p((x_n)) < \infty$ for all $(x_n) \in l^\infty$. This follows, since

$$p((x_n)) = \limsup_{n\to\infty} \left| \frac{1}{n} \sum_{i=1}^n x_i \right| \leq \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^n \sup_n |x_n| = \sup_n |x_n| < \infty.$$

Next, we need to show that for all $y \geq 0$, we have

$$p(y(x_n)) = yp((x_n)).$$

Notice first that

$$y(x_n) = (yx_n),$$

and so we have

$$p(y(x_n)) = p((yx_n)) = \limsup_{n\to\infty} \left| \frac{1}{n} \sum_{i=1}^n yx_i \right| = y \limsup_{n\to\infty} \left| \frac{1}{n} \sum_{i=1}^n x_i \right|$$
$$= yp((x_n)).$$

Finally, we need to show that

$$p((x_n) + (y_n)) \leq p((x_n)) + p((y_n)).$$

Notice that

$$(x_n) + (y_n) = (x_n + y_n),$$

$$p((x_n + y_n)) = \limsup_{n\to\infty} \left| \frac{1}{n} \sum_{i=1}^n (x_i + y_i) \right| \leq \limsup_{n\to\infty} \left| \frac{1}{n} \sum_{i=1}^n x_i \right| + \limsup_{n\to\infty} \left| \frac{1}{n} \sum_{i=1}^n y_i \right| = p((x_n)) + p((y_n)).$$

So we get that p is a sublinear functional.

Now, define a linear functional L on $c \subset l^\infty$ where

$$L((x_n)) = \lim x_n.$$

This is clearly linear by properties of limits, and furthermore we have

$$L((x_n)) = \lim x_n = \lim \frac{1}{n} \sum_{i=1}^n x_i \leq \limsup \left| \frac{1}{n} \sum_{i=1}^n x_i \right| = p((x_n))$$

by Cesàro mean properties. Hence, we can use Hahn-Banach to find a linear functional which extends L to the entire space, denoted by φ. That is, we have

$$\varphi \leq p,$$

$$\varphi|_c = L.$$

We now need to show φ satisfies the desired properties. First, we show (1). That is, we need to show that

$$\varphi(S(x_n)) = \varphi((x_n))$$

for all $(x_n) \in l^\infty$. This is equivalent to showing that

$$|\varphi(S(x_n)) - \varphi((x_n))| = |\varphi(S(x_n) - (x_n))| = 0.$$

Notice we have

$$|\varphi(S(x_n) - (x_n))| \leq p(S(x_n) - (x_n)),$$

since we have that $p(-x) = p(x)$ by construction. Notice as well that

$$p(S(x_n) - (x_n)) = \limsup_{n\to\infty} \left| \frac{1}{n} \sum_{i=1}^{n} (x_{i+1} - x_i) \right| = \limsup_{n\to\infty} \left| \frac{x_n - x_1}{n} \right| \leq \limsup_{n\to\infty} \frac{2||x_n||_\infty}{n} = 0.$$

So in other words, we have $\varphi(S(x_n)) = \varphi((x_n))$.

Next, we need to show that if $(x_n) \geq 0$, we have $\varphi((x_n)) \geq 0$. Take $(x_n) \geq 0$, and write it as $(x_n) = c \cdot (y_n)$, where $y_n \in [0, 1]$. Now notice that

$$1 - \varphi((y_n)) = \varphi(1 - (y_n)) \leq \limsup_{n\to\infty} \left| \frac{1}{n} \sum_{i=1}^{n} (1 - y_i) \right| \leq 1,$$

and so

$$0 \leq \varphi((y_n)),$$

and then using linearity, we get it holds for all $(x_n) \geq 0$.

Next, we need to show that

$$\liminf x_n \leq \varphi(x) \leq \limsup x_n.$$

Recall that

$$\limsup x_n = \inf_{n\geq 1} \sup_{k\geq n} x_k.$$

Take α such that

$$\limsup x_n = \inf_{n\geq 1} \sup_{k\geq n} x_k < \alpha.$$

Then we have that there is a point N so that for all $n \geq N$,

$$x_n < \alpha.$$

Applying the shift $S^{(N)}$ to x, we have

$$\varphi(\alpha - S^{(n)}(x_n)) = \alpha\varphi(1) - \varphi(S^{(n)}(x_n)) = \alpha - \varphi((x_n)).$$

Notice as well that $\alpha - S^{(N)}((x_n))$ is a positive bounded sequence, so we get that

$$0 \leq \alpha - \varphi((x_n)) \implies \varphi(x_n) \leq \alpha.$$

Since this applies for all $\alpha > \limsup x_n$, we get that

$$\varphi(x_n) \leq \limsup x_n.$$

Analogously, take

$$\alpha < \liminf x_n = \sup_{n\geq 1} \inf_{k\geq n} x_k.$$

Then there exists a point N such that for all $n \geq N$,

$$\alpha < x_n.$$

Hence, noting that $S^{(N)}(x_n) - \alpha$ is a positive bounded sequence, we have

$$0 \leq \varphi((x_n)) - \alpha \implies \alpha \leq \varphi((x_n)),$$

and so we have that, since this applies for all α,

$$\liminf x_n \leq \varphi((x_n)).$$

$\square$

Problem 57. Let X be a compact Hausdorff topological space. For $x \in X$, define $\text{ev}_x : C(X) \to F$ by $\text{ev}_x(f) = f(x)$.

(1) Prove that $\text{ev}_x \in C(X)^*$ and find $||\text{ev}_x||$.
(2) Show that the map $\text{ev} : X \to C(X)^*$ given by $x \mapsto \text{ev}_x$ is a homeomorphism onto its image, where the image has the relative weak* topology.

Proof. (1) Recall that $C(X)^* = \mathcal{L}(C(X), F)$. We need to show that ev_x is linear and is bounded. We have that

$$\text{ev}_x(f + g) = (f + g)(x) = f(x) + g(x),$$

and for α any scalar,

$$\text{ev}_x(\alpha f) = \alpha f(x) = \alpha \text{ev}_x(f).$$

So it is indeed linear. Next, notice that

$$||\text{ev}_x|| = \sup\{||\text{ev}_x(f)|| \ : \ f \in C(X),\ ||f|| \leq 1\} = \sup\{||f(x)|| \ : \ ||f|| \leq 1,\ f \in C(X)\}.$$

Since $||f|| \leq 1$, this means that $-1 \leq f \leq 1$ on X. Hence, we see that $||\text{ev}_x|| = 1$ (just take a continuous function which is 1). So it is bounded and linear, hence $\text{ev}_x \in C(X)^*$.

(2) We check first that $\text{ev} : X \to C(X)^*$ is injective. We can equivalently show that if $x \neq y$, there is a $f \in C(X)$ so that $\text{ev}_x(f) \neq \text{ev}_y(f)$. The space is Hausdorff, so we can find open neighborhoods which separate the points. Urysohn's Lemma then gives us that we can find a continuous function f so that $\text{ev}_x(f) \neq \text{ev}_y(f)$. So it is injective. It's clearly surjective onto its image, so we have it's a bijection.

We now need to check it's a homeomorphism. Since it's with the relative weak* topology, let (x_n) be a net converging to some point $x \in X$. The weak* topology says that for every $f \in C(X)$, we have that $f(x_n) \to f(x)$; in other words, $\text{ev}_{x_n} \to \text{ev}_x$ in the weak* topology. So ev is continuous.

Finally, we use **Proposition 4.28**. We have that X is compact, $C(X)^*$ is Hausdorff, so a continuous bijection is a homeomorphism.

$\square$

Problem 58. Suppose X and Y are Banach spaces and $T : X \to Y$ a linear transformation.

(1) Show that if $T \in \mathcal{L}(X, Y)$, then T is weak-weak continuous.
(2) Show that if T is norm-weak continuous, then $T \in \mathcal{L}(X, Y)$.
(3) Show that if T is weak-norm continuous, then T has finite rank.

Proof. (1) This is by the quiz. We need to show that if $x_n \to x$ weakly (for all $\varphi \in X^*$, $\varphi(x_n) \to x$), then $T(x_n) \to T(x)$ weakly (for all $\gamma \in T^*$, $\gamma(T(x_n)) \to x_n$). Take $\gamma \in Y^*$ arbitrarily. Then we have that $\gamma \circ T : X \to F$, and furthermore it is bounded since both are bounded. Hence, $\gamma \circ T \in X^*$. Since $x_n \to x$ weakly, we have $\gamma \circ T(x_n) \to \gamma \circ T(x)$. Since γ was chosen arbitrarily, we have that $T(x_n) \to T(x)$ for all $\gamma \in Y^*$; in other words, $T(x_n) \to T(x)$ weakly.

(2) According to the quiz solutions, we use the Closed Graph theorem. Suppose $(x_n, T(x_n)) \to (x, y)$ in norm, then we wish to show that $x = T(x)$ for some $x \in X$. Since T is norm-weak continuous, we have that $T(x_n) \to T(x)$ weakly. Since $T(x_n) \to y$ in norm, we get in particular that $T(x_n) \to y$ weakly; to see this, take any $\varphi \in Y^*$. Then we have

$$||\varphi(T(x_n) - y)|| \leq C||T(x_n) - y|| \to 0,$$

so

$$\varphi(T(x_n)) \to \varphi(y).$$

Hence, $T(x_n) \to y$ weakly as well. The weak topology is Hausdorff, so we have that $y = T(x)$, and hence the graph is closed. The Closed Graph theorem then gives that T is bounded.

(3) Using **Proposition 5.15**, we get that, since T is weak-norm continuous, there exists $L_1, \ldots, L_n \in X^*$ and $C > 0$ so that

$$||T(x)|| \leq C \sum_{i=1}^{n} ||L_i(x)||$$

So in particular, we get that $x \in \bigcap_{i=1}^n \ker(L_i)$ implies that $x \in \ker(T)$, so $\bigcap_{i=1}^n \ker(L_i) \subset \ker(T)$. So we see that $\operatorname{rank}(T) \leq n$. In other words, the rank is finite.

□

Problem 59. Consider the space $L^2(T) := L^2(\mathbb{R}/\mathbb{Z})$ of $\mathbb{Z}$-periodic functions $\mathbb{R} \to \mathbb{C}$ such that

$$\int_{[0,1]} |f|^2 < \infty.$$

Define

$$\langle f, g \rangle := \int_{[0,1]} f\overline{g}.$$

(1) Prove that $L^2(\mathbb{R}/\mathbb{Z})$ is a Hilbert space.
(2) Show that the subspace $C(T) \subset L^2(T)$ of continuous $\mathbb{Z}$ periodic functions is dense.
(3) Prove that

$$\{e_n(x) := \exp(2\pi i n x) \ : \ n \in \mathbb{Z}\}$$

is an orthonormal basis for $L^2(T)$.
(4) Define $\mathcal{F} : L^2(T) \to l^2(\mathbb{Z})$ by

$$\mathcal{F}(f)_n := \langle f, e_n \rangle_{L^2(T)} = \int_0^1 f(x) \exp(-2\pi i n x) dx.$$

Show that if $f \in L^2(T)$ and $\mathcal{F}(f) \in l^1(\mathbb{Z})$, then $f \in C(T)$. In other words, f is a.e. equal to a continuous function.

Proof. (1) To show it's a Hilbert space, we need to show that this is a vector space, this defines an inner product, and that with respect to this inner product it is complete. We first check that this is a vector space over $\mathbb{R}$. Most of these properties are clear; after showing that it's closed under addition and scalar multiplication, we have that $f + (g + h) = (f + g) + h$, $f + g = g + f$, $0 \in L^2(T)$, $-f \in L^2(T)$, for $a, b \in \mathbb{R}$ we have $a(bf) = (ab)f$, $1f = f$, $a(f + g) = af + ag$, $(a + b)f = af + bf$.

To get closure under addition, we can note that

$$(f - g)^2 = f^2 - 2fg + g^2 \geq 0 \implies f^2 + g^2 \geq 2fg,$$

so taking absolute values gives

$$2|f||g| \leq |f|^2 + |g|^2.$$

Hence, we have

$$\int |f+g|^2 = \int |f|^2 + 2\int |f||g| + \int |g|^2 \leq 2\int |f|^2 + 2\int |g|^2 < \infty,$$

so $f + g \in L^2(T)$. We also have

$$\int |af|^2 = |a|^2 \int |f|^2 < \infty$$

for all $a \in \mathbb{R}$, so it is closed under scalars. Hence, $L^2(T)$ is a vector space.

We now check that this defines an inner product.

(a) We have

$$\langle af + bg, h\rangle = \int_{[0,1]} (af+bg)\overline{h} = a\int_{[0,1]} f\overline{h} + b\int_{[0,1]} g\overline{h} = a\langle f,h\rangle + b\langle g,h\rangle,$$

via properties of the integral.

(b) We have

$$\langle g, f\rangle = \int_{[0,1]} g\overline{f} = \overline{\int_{[0,1]} f\overline{g}} = \overline{\langle f,g\rangle}.$$

(c) We see

$$\langle f, f\rangle = \int_{[0,1]} f\overline{f} = \int_{[0,1]} |f|^2 \in (0,\infty)$$

for f non-zero a.e.

Hence, it is an inner product. Next, we need to show that it's complete with respect to the norm given by

$$||f|| = \sqrt{\langle f, f\rangle}.$$

We follow the proof of **Theorem 6.6** in Folland. By **Theorem 5.1**, it suffices to show that every absolutely convergent series in $L^2(T)$ converges. So, take

$$\sum_{i=1}^{\infty} ||f_i|| < \infty.$$

Let $F_n = \sum_1^n |f_i|$, $F = \sum_1^\infty |f_i|$. Then we see that

$$\|F_n\| = \left\|\sum_1^n |f_i|\right\| \leq \sum_1^n \|f_i\| < \sum_1^\infty \|f_i\|.$$

By the monotone convergence theorem, we get

$$\int F^2 = \lim \int F_n^2 < \infty.$$

So $F \in L^2(T)$, implying that $\sum_1^\infty f_i$ converges since $F(x) < \infty$ a.e. Letting $G = \sum_1^\infty f_i$, we get $|G| \leq F$, and so $G \in L^2(T)$. Using the fact that

$$|G - \sum_1^n f_i|^2 \leq (2F)^2 \in L^1,$$

we can use dominated convergence theorem to get

$$\|F - \sum_1^n f_i\|^2 \to 0,$$

and so we have satisfied the conditions of **Theorem 5.1**. Hence, it is complete.

(2) We break it up into steps.
Step 1: (Follow proof of **Proposition 6.7** in Folland.) We show that simple functions are dense in $L^2(T)$. Let

$$g = \sum_{i=1}^{n} a_i \chi_{E_i},$$

where $\mu(E_i) < \infty$ for all i, $E_i \subset T$ measurable subsets. It's clear that any such g is in $L^2(T)$. We then need to show that we can find a sequence (f_n) such that $f_n \to f$ in $L^2(T)$. By the theorem from class, we can choose $f_n \nearrow f$ so that $|f_n| \le |f|$ for all n, $f_n \to f$ a.e. We get that

$$|f_n - f|^2 \le 4|f|^2,$$

since

$$|f_n - f|^2 \le (|f_n| + |f|)^2 = |f_n|^2 + 2|f_n||f| + |f|^2 \le |f|^2 + 2|f|^2 + |f|^2 = 4|f|^2.$$

By the dominated convergence theorem, we then have that

$$\int |f_n - f|^2 \to 0,$$

or in other words,

$$\|f_n - f\|^2 \to 0.$$

Hence,

$$\|f_n - f\| \to 0.$$

Step 2: (Follow the proof of **Theorem 7.9** in Folland.) We show that we can approximate characteristic functions using continuous functions. Once we have done this, as in prior homeworks, we can deduce that $C(T)$ is dense in $L^2(T)$.

Let E be any Borel set with $\mu(E) < \infty$. Using the regularity of the Lebesgue measure on this, fixing $\epsilon > 0$, we get that we can find an open U such that $E \subset U$ and a compact K such that $K \subset U$ so that $\mu(E - K) < \epsilon$. By Urysohn's lemma, we can find continuous f so that $\chi_K \le f \le \chi_U$, and so

$$\|\chi_E - f\| = \sqrt{\int |\chi_E - f|^2} \le \sqrt{\int |\chi_U - \chi_K|^2} \le \sqrt{\epsilon}.$$

This tells us that we can approximate any simple function arbitrarily well with a continuous function, and so therefore the continuous functions are dense in $L^2(T)$.

(3) (We follow **Theorem 8.20** in Folland) We check that these are orthogonal in $L^2(T)$. We have

$$\langle e_n, e_m \rangle = \int_{[0,1]} e^{2\pi i(n-m)x} dx.$$

Let $u = 2\pi i(n-m)x$, we have $du = 2\pi i(n-m)dx$, so

$$\int \frac{e^u}{2\pi i(n-m)} du = \frac{e^{2\pi i(n-m)x}}{2\pi i(n-m)}\Bigg|_{x=0}^{1} = \frac{e^{2\pi(n-m)i} - 1}{2\pi i(n-m)}.$$

Since $m, n \in \mathbb{Z}$, we have that

$$e^{2\pi i(n-m)} = 1.$$

Hence, for $n \ne m$, we get that

$$\langle e_n, e_m \rangle = 0,$$

and for $n = m$, we see that we get

$$\langle e_n, e_n \rangle = 1.$$

So these are indeed orthonormal.

We now check that this is an algebra. However, from our work before (**Problem 50**), we can use DeMoivre's to deduce that this is indeed an algebra. We also can deduce that this separates points from the solution as well. We have T is compact, and Stone-Weierstrass gives that this is dense in the uniform norm. Notice that being dense in the uniform norm also gives being dense in the L^2 norm. Fix $\epsilon > 0$, $g \in C(T)$. Since we have denseness in the uniform norm, we can find $f \in \mathcal{A}$ (our the span of the $e_n(x)$ will be the algebra denoted by $\mathcal{A}$) so that

$$\|f - g\|_\infty < \epsilon,$$

then we have that

$$\|f - g\|^2 = \int_{[0,1]} |f - g|^2 < \epsilon^2,$$

so that

$$\|f - g\| < \epsilon.$$

So we have that $\mathcal{A}$ is dense in $C(T)$, which is dense in $L^2(T)$, and so $\mathcal{A}$ is dense in $L^2(T)$. Thus, it's an orthonormal basis by the theorem in the class notes.

(4) We have

$$\mathcal{F}(f) = (\hat{f}_n),$$

where

$$\hat{f}_n = \mathcal{F}(f)_n = \int_{[0,1]} f(x) \exp(-2\pi inx) dx.$$

Since $\mathcal{F}(f) \in l^1(\mathbb{Z})$, we have that

$$\sum |\hat{f}_n| < \infty.$$

Furthermore, as Folland defines, we have that

$$\sum_{n \in \mathbb{Z}} \hat{f}_n e_n$$

is the Fourier series of f, and this property says that the Fourier series of f converges. By the prior part, we have that the Fourier coefficients $\hat{f}$ are unique to f (up to a.e. equivalence), and so any function g which shares the same Fourier coefficients will be equal to f a.e. Let

$$S_N := \sum_{n=-N}^{N} \hat{f}_n e_n.$$

We have that S_N is continuous, and from the condition that $\mathcal{F}(f) \in l^1(\mathbb{Z})$ and by Weierstrass M-test, we see that S_N converges uniformly to a function g which must also be continuous. Furthermore, we have that the Fourier coefficients of g and f are the same, and so we have that f and g are equal a.e.

□

Problem 60. Suppose H is a Hilbert space, $E \subset H$ is an orthonormal set, and $\{e_1, \ldots, e_n\} \subset E$. Prove the following assertions.

(1) If

$$x = \sum_{i=1}^{n} c_i e_i,$$

then

$$c_i = \langle x, e_i \rangle.$$

(2) The set E is linearly independent.

(3) For every $x \in H$,
$$\sum_{i=1}^{n} \langle x, e_i \rangle e_i$$
is the unique element of $\text{span}\{e_1, \ldots, e_n\}$ minimizing the distance to x.

(4) (Bessel's Inequality) For every $x \in H$,
$$||x||^2 \geq \sum_{i=1}^{n} |\langle x, e_i \rangle|^2.$$

(5) If H is separable, then E is countable.

(6) The set E can be extended to an orthonormal basis for H.

(7) If E is an orthonormal basis, then the map $H \to l^2(E)$ given by $x \mapsto (\langle x, \cdot \rangle : E \to \mathbb{C})$ is a unitary isomorphism of Hilbert spaces.

Proof. (1) Since it's orthonormal, we have that $\langle e_i, e_j \rangle = \delta_{ij}$. Notice that by linearity in the first component, we have
$$\langle x, e_i \rangle = \left\langle \sum_{j=1}^{n} c_j e_j, e_i \right\rangle = \sum_{j=1}^{n} c_j \langle e_j, e_i \rangle = c_i \langle e_i, e_i \rangle = c_i ||e_i|| = c_i.$$

(2) To show E is linearly independent, we need to show that any finite subset $S \subset E$ is linearly independent. Taking such a finite subset, we can represent it as $S = \{e_1, \ldots, e_n\}$. We want to then show that
$$\sum a_i e_i = 0 \implies a_i = 0 \text{ for all } i.$$
Since it's orthonormal, we have the Pythagorean theorem applies to give
$$\left\| \sum a_i e_i \right\|^2 = \sum |a_i|^2 \|e_i\|^2 = \sum |a_i|^2 = 0.$$
Hence, we must have that $a_i = 0$ for all i. Since the choice of finite subset was arbitrary, we get that E is linearly independent.

(3) Write $M = \text{span}\{e_1, \ldots, e_n\}$. Then we have from the class notes that
$$H = M \oplus M^{\perp}.$$
Hence, we have that for all $x \in H$, we can write
$$x = y + z,$$
where $y \in M$, $z \in M^{\perp}$. Since $y \in M$, we have
$$y = \sum a_i e_i,$$
and so we can write
$$x = \sum a_i e_i + z.$$
Now use the same strategy as before; we have
$$\langle x, e_j \rangle = \left\langle \sum a_i e_i + z, e_j \right\rangle = \sum a_i \langle e_i, e_j \rangle + \langle z, e_j \rangle = a_j.$$
Hence, the unique element minimizing distance between M and x is the desired quantity.

(4) (Folland Theorem 5.26) Notice that, by the Pythagorean theorem, we get
$$0 \leq \left\| x - \sum_{i=1}^{n} \langle x, e_i \rangle e_i \right\|^2 = \|x\|^2 - 2\text{Re} \left\langle x, \sum_{i=1}^{n} \langle x, e_i \rangle e_i \right\rangle + \left\| \sum_{i=1}^{n} \langle x, e_i \rangle e_i \right\|^2$$

$$= \|x\|^2 - 2\sum_{i=1}^n |\langle x, e_i\rangle|^2 + \sum_{i=1}^n |\langle x, e_i\rangle|^2$$
$$= \|x\|^2 - \sum_{i=1}^n |\langle x, e_i\rangle|^2.$$

(5) (Folland Theorem 5.29) Since H is separable, we can show that it has a countable orthonormal basis. With this, we have that every orthonormal basis is countable, and by the theorem from class we have that every orthonormal set is contained in an orthonormal basis by maximality (we implicitly use (6) here), hence at most countable.

To see that the orthonormal basis has to be countable, let $S = \{x_n\}$ be a countable dense subset of H. We construct another set as follows: Let $y_1 = x_1$, $S_1' = \{x_1\}$. Let $S_1'' = \text{span}(y_1)^c \cap S$. After reordering, we can write this as $S_1'' = \{x_2, \ldots\}$. Let $y_2 = x_2$, $S_2' = \{y_1, y_2\}$, $S_2'' = \text{span}(y_1, y_2)^c \cap S$, reorder, and continue. We then get a new countable set $S' = \bigcup S_i' = \{y_n\}$. We can then apply Graham-Schmidt to this new set to get an orthonormal sequence $\{u_n\}$ which is dense, and so therefore is a basis. Hence, the orthonormal basis has to be countable.

(6) If E is an orthonormal basis, then we are done. Else, take the collection of all orthonormal sets, ordered by inclusion, and notice that if we have a chain of orthonormal sets $E_1 \subset E_2 \subset \cdots$, then $E = \bigcup E_i$ is an orthonormal set. It's clear that all $a \in E$ have $||a|| = 1$, since $a \in E_i$ for some i large enough. Furthermore, taking $a, b \in E$, $a \neq b$, we have that $a, b \in E_n$ for some n large enough, and so $a \perp b$. Furthermore, it's clear we have that it's maximal, so by the theorem in the class notes we have E is an orthonormal basis. So if E is not an orthonormal basis, it is contained in one, and hence it can be extended.

(7) (Folland Proposition 5.30/Exercise 5.55) We check that the map is linear first. Let $T : H \to l^2(E)$ be the map which sends $T(x) = \langle x, \cdot\rangle$. Then we need to show that $T(ax + y) = aT(x) + T(y)$. We have

$$T(ax + y) = \langle ax + y, \cdot\rangle = a\langle x, \cdot\rangle + \langle y, \cdot\rangle = aT(x) + T(y).$$

By Parseval's identity, we have

$$||x||^2 = \sum_{a\in A} |\langle x, e_i\rangle|^2$$

for all $x \in H$. Thus, we have that it's an isometry.

Taking $g \in l^2(E)$, we have that

$$\sum_{e\in E} |g(e)|^2 < \infty,$$

and the Pythagorean applies to give that the partial sums of $\sum_{e\in E} g(e)e$ are Cauchy, so $x = \sum_{e\in E} g(e)e$ is in H and $T(x) = g$. That is, the map is surjective. Isometry gives us that it is injective, so it is an invertible linear map.

Next, we need to show that it preserves the inner product. We first show the polarization identity.

Let $x, y \in H$. Then we see that

$$||x + y||^2 = \langle x + y, x + y\rangle = \langle x, x + y\rangle + \langle y, x + y\rangle = ||x||^2 + \langle x, y\rangle + \langle y, x\rangle + ||y||^2$$
$$= ||x||^2 + 2\text{Re}(\langle x, y\rangle) + ||y||^2,$$
$$||x - y||^2 = \langle x - y, x - y\rangle = \langle x, x - y\rangle - \langle y, x - y\rangle = ||x||^2 - \langle x, y\rangle - \langle y, x\rangle + ||y||^2$$
$$= ||x||^2 - 2\text{Re}(\langle x, y\rangle) + ||y||^2,$$
$$||x + iy||^2 = \langle x + iy, x + iy\rangle = \langle x, x + iy\rangle + i\langle y, x + iy\rangle = ||x||^2 - i\langle x, y\rangle + i\langle y, x\rangle + ||y||^2$$

$$= ||x||^2 + 2\mathrm{Im}(\langle x, y\rangle) + ||y||^2,$$

$$||x - iy||^2 = \langle x - iy, x - iy\rangle = \langle x, x - iy\rangle - i\langle y, x - iy\rangle = ||x||^2 + i\langle x, y\rangle - i\langle y, x\rangle + ||y||^2$$

$$= ||x||^2 - 2\mathrm{Im}(\langle x, y\rangle) + ||y||^2.$$

Combining this together, we get

$$||x + y||^2 - ||x - y||^2 + i||x + iy||^2 - i||x - iy||^2 = 4\langle x, y\rangle,$$

as desired. Now notice that

$$\langle T(x), T(y)\rangle = \frac{1}{4}\left(||T(x) + T(y)||^2 - ||T(x) - T(y)||^2 + i||T(x) + iT(y)||^2 - i||T(x) - iT(y)||^2\right)$$

$$= \frac{1}{4}\left(||T(x + y)||^2 - ||T(x - y)||^2 + i||T(x + iy)||^2 - i||T(x - iy)||^2\right)$$

$$= \frac{1}{4}\left(||x + y||^2 - ||x - y||^2 + i||x + iy||^2 - i||x - iy||^2\right) = \langle x, y\rangle$$

by isometry properties, so we have that it is unitary.

□

Remark. Thomas O'Hare was a collaborator.

Problem 61. Let X be a LCH space and suppose $\varphi : C_0(X) \to \mathbb{C}$ is a linear functional such that $\varphi(f) \geq 0$ whenever $f \geq 0$. Prove that φ is bounded.

Proof. We follow the hint; that is, we wish to show that

$$\{\varphi(f) \ : \ 0 \leq f \leq 1, f \in C_0(X)\}$$

is bounded. Recall that $C_0(X)$ is the collection of continuous functions which vanish at infinity; for every $\epsilon > 0$, $\{f \geq \epsilon\}$ is compact. By the hint in the class, we proceed by contradiction: that is, φ is *not* bounded. Then we can construct a sequence $(f_n) \subset C_0(X)$ so that $0 \leq f_n \leq 1$ for all n, but $\varphi(f_n) \to \infty$. We use the Weierstrass M-test here. Choose $\varphi(f_n) > n^2$. Then we get that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} f_n$$

converges uniformly, since $0 \leq f_n \leq 1$, so taking $M_n = \frac{1}{n^2}$ we have a convergent series and so the M-test applies. Since it's a uniformly convergent series of continuous functions, we get that

$$f = \sum_{n=1}^{\infty} \frac{1}{n^2} f_n$$

is a continuous function. We check now that $f \in C_0(X)$. Fix $\epsilon > 0$ and examine the set

$$\{x \in X \ : \ f \geq \epsilon\} = \left\{ x \in X \ : \ \sum_{n=1}^{\infty} \frac{1}{n^2} f_n \geq \epsilon \right\}.$$

For fixed ϵ, there exists an N large enough so that

$$\frac{1}{N^2} < \epsilon.$$

Since $0 \leq f_n \leq 1$ for each n, this means that none of the x contribute with regards to f_n for n sufficiently large. So we have

$$\{x \in X \ : \ f \geq \epsilon\} \subset \bigcup_{n=1}^{N-1} \{x \in X \ : \ f_n \geq \epsilon n^2\}.$$

Now, notice that f is continuous, so this means that the set on the left is closed. Since it's a finite union of compact sets on the right, this means it is compact. We have a closed subset of a compact set, and so this must be compact. Hence, $f \in C_0(X)$.

Normalizing f to get

$$g(x) := \frac{6}{\pi^2} f(x),$$

we have $g \in C_0(X)$ and $0 \leq g \leq 1$. We have that

$$\varphi(g) = \varphi\left(\frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} f_n\right).$$

Since everything is positive, we have

$$\varphi(g) \geq \frac{6}{\pi^2} \sum_{n=1}^{N} \frac{1}{n^2} \varphi(f_n).$$

We chose f_n so that $\varphi(f_n) > n^2$, so we have

$$\varphi(g) \geq \frac{6}{\pi^2} \sum_{n=1}^{N} \frac{1}{n^2} \varphi(f_n) > \frac{6}{\pi^2} \sum_{n=1}^{N} 1.$$

This holds for all N, but this means that $\varphi(g) = \infty$. This is a contradiction, and so we must have that this set is bounded.

□

Problem 62. Let X be a LCH space, $K \subset X$ compact, and $U_1, \ldots, U_n$ open sets such that

$$K \subset \bigcup_{i=1}^{n} U_i.$$

Show that there exists $g_1, \ldots, g_n \in C_c(X)$ such that $g_i \prec U_i$ for all i and $\sum_1^n g_i = 1$ on K.

Proof. Choose $x \in K$ such that there are compact neighborhoods $N_x \subset U_j$ for some j. We have that

$$K \subset \bigcup N_x^{\text{o}},$$

so by compactness we get

$$K \subset \bigcup_{i=1}^{m} N_{x_i}^{\text{o}} \subset \bigcup_{i=1}^{m} N_{x_i}.$$

Taking $F_j = \bigcup_{l=1}^{k} N_{x_l}$, where $N_{x_l} \subset U_j$, we get that the F_j are compact (since they are finite unions of compact things), and moreover we have $F_j \subset U_j$ for each $j \in \{1, \ldots, n\}$. By LCH Urysohn, we have that we can find continuous h_j such that $h_j = 1$ on F_j and $h_j = 0$ outside of a compact subset $V \subset U_j$. Moreover, this gives us that $h_j \prec U_j$ for each j. Notice that we have that

$$\sum_{i=1}^{n} h_j \geq 1$$

on K. Use LCH Urysohn to define f so that $f = 1$ on K and $\overline{\text{supp}(f)} \subset \{x \,:\, \sum h_i > 0\}$. Letting $h_{n+1} = 1 - f$, we have that $\sum_{i=1}^{n+1} g_i > 0$ on all X. Let

$$g_j = \frac{h_j}{\sum_{i=1}^{n+1} h_i}$$

for $j \in \{1, \ldots, n\}$.s Then we have that $g_j \prec U_j$ still, and furthermore we have that

$$\sum_{j=1}^{n} g_j = 1$$

on K by construction. □

Problem 63. Suppose X is a LCH space, μ is a σ-finite Radon measure on X, and E is a Borel set. Prove that for every $\epsilon > 0$, there is an open set U and a closed set F with $F \subset E \subset U$ such that

$$\mu(U - F) < \epsilon.$$

Proof. Using disjointification, write E as

$$E = \bigsqcup_{n=1}^{\infty} E_n,$$

where $\mu(E_n) < \infty$. By the class notes (or **Proposition 7.5** in Folland), we have that we can find open U_n so that

$$\mu(U_n) < \mu(E_n) + \epsilon 2^{-n-1}.$$

Write

$$U = \bigcup_{n=1}^{\infty} U_n.$$

U is open, and furthermore $E \subset U$,

$$\mu(E - U) \leq \sum_{n=1}^{\infty} \mu(U_n - E_n) < \frac{\epsilon}{2}.$$

We have that E^c is also a Borel set, and so we can write

$$E^c = \bigsqcup_{n=1}^{\infty} G_n,$$

where $\mu(G_n) < \infty$. For each G_n, find open V_n so that

$$\mu(V_n) < \mu(G_n) + \epsilon 2^{-n-1}.$$

Write

$$V = \bigcup_{n=1}^{\infty} V_n.$$

Then we have that

$$\mu(E^c - V) < \frac{\epsilon}{2}.$$

Letting $F = V^c$, we have that F is closed and $F \subset E$. Notice as well that

$$\mu(U - F) = \mu(U - E) + \mu(E - F) = \mu(U - E) + \mu(V - E^c) < \epsilon.$$

□

Problem 64. Suppose X is an LCH space and $\varphi \in C_0(X)^*$. Prove that there are finite Radon measures $\mu_0, \mu_1, \mu_2, \mu_3$ on X such that

$$\varphi(f) = \sum_{k=0}^{3} i^k \int f d\mu_k$$

for all $f \in C_0(X)$.

Proof. Consider the case where $\varphi \in C_0(X, \mathbb{R})^*$. By the class notes, we have that there are positive linear functionals $\varphi_\pm \in C_0(X, \mathbb{R})^*$ such that $\varphi = \varphi_+ - \varphi_-$. We also get that positive linear functionals are of the form

$$\varphi_\pm = \int \cdot d\mu_\pm,$$

where $\mu_\pm$ are finite Radon measures.

Examining the case where $\varphi \in C_0(X)^*$, we take $f \in C_0(X)$ and notice that it can be written as $f = \text{Re}(f) + i\text{Im}(f)$, where $\text{Re}(f), \text{Im}(f) \in C_0(X, \mathbb{R})$. Using linearity, we have

$$\varphi(f) = \varphi(\text{Re}(f)) + i\varphi(\text{Im}(f)).$$

Hence, we can write $\varphi_1 = \varphi \circ \text{Re}$ and $\varphi_2 = \varphi \circ \text{Im}$ to write this as

$$\varphi(f) = \varphi_1(f) + i\varphi_2(f).$$

Now, we notice that $\varphi_1, \varphi_2 \in C_0(X, \mathbb{R})^*$, and so by the previous discussion we get that we can decompose them as

$$\varphi_1 = \int \cdot d\mu_0 - \int \cdot d\mu_2,$$
$$\varphi_2 = \int \cdot d\mu_1 - \int \cdot d\mu_3.$$

Expanding this, we have

$$\varphi(f) = \int f d\mu_0 - \int f d\mu_2 + i \int f d\mu_1 - i \int f d\mu_3.$$

Rearranging terms, we have

$$\varphi(f) = \sum_{k=0}^{3} i^k \int f d\mu_k,$$

as desired.

□

Remark. Thomas O'Hare was a collaborator.

Problem 65. Suppose μ is a positive measure on $(X, \mathcal{M})$ and ν is a signed measure on $(X, \mathcal{M})$.

(1) Prove that the following are equivalent:
  (a) $\nu \perp \mu$
  (b) $|\nu| \perp \mu$
  (c) $\nu_+ \perp \mu$ and $\nu_- \perp \mu$
(2) Prove that the following are equivalent:
  (a) $\nu \ll \mu$
  (b) $|\nu| \ll \mu$
  (c) $\nu_+ \ll \mu$ and $\nu_- \lll \mu$.

Proof. (1) (a) $\implies$ (b): Assume $\nu \perp \mu$. Then we have that we can write

$$X = E \sqcup F,\ \mu(E) = \nu(F) = 0.$$

We can write $\nu = \nu_+ - \nu_-$. Hahn decomposition gives us

$$X = P \sqcup N,\ \nu_+(N) = \nu_-(P) = 0.$$

Notice that $|\nu| = \nu_+ + \nu_-$, where $\nu_+(E) = \nu(E \cap P)$, $\nu_-(E) = -\nu(E \cap N)$. Notice now that

$$|\nu|(F) = \nu_+(F) + \nu_-(F) = \nu(F \cap P) - \nu(F \cap N).$$

Since $\nu(F) = 0$ (moreover, F is ν-null), we have $\nu(F \cap P) = 0$, $\nu(F \cap N) = 0$, so

$$|\nu|(F) = 0.$$

Hence, we have $|\nu| \perp \mu$.
(b) $\implies$ (c): We can write

$$X = E \sqcup F,\ \mu(E) = |\nu|(F) = 0.$$

Again, write

$$X = P \sqcup N,\ \nu_+(N) = \nu_-(P) = 0.$$

Write

$$X = (E \cap N) \sqcup (E \cap P) \sqcup (F \cap N) \sqcup (F \cap P).$$

Notice that

$$0 \leq \mu(E \cap P) \leq \mu(E) = 0,$$

so $\mu(E \cap P) = 0$. Similarly, we have

$$\nu_+(E \cap N) + \nu_+(F \cap N) + \nu_+(F \cap P) = 0,$$

since each respective component is 0. Thus, $\nu_+ \perp \mu$. An analogous argument shows that $\nu_- \perp \mu$.
(c) $\implies$ (a): Since $\nu_+ \perp \mu$, we can write

$$X = E \sqcup F,\ \mu(E) = \nu_+(F) = 0.$$

Similarly, write

$$X = G \sqcup H,\ \mu(H) = \nu_-(G) = 0.$$

Intersecting these gives

$$X = (E \cap G) \sqcup (E \cap H) \sqcup (F \cap G) \sqcup (F \cap H).$$

Let $P = F \cap G$, $N = (E \cap G) \sqcup (E \cap H) \sqcup (F \cap H)$. Write as well

$$X = K \sqcup L, \ \nu_+(L) = \nu_-(K) = 0.$$

Then we have

$$\nu(P) = \nu_+(F \cap G \cap K) - \nu_-(F \cap G \cap L).$$

Since $\nu_+(F) = 0$, $\nu_+(F \cap G \cap K) \le \nu_+(F)$, we have that $\nu_+(F \cap G \cap K) = 0$. Likewise, we get $\nu_-(F \cap G \cap L) = 0$. Hence, $\nu(P) = 0$.

Similarly, we see that

$$\mu(N) = \mu(E \cap G) + \mu(E \cap G) + \mu(F \cap H) \le \mu(E) + \mu(E) + \mu(H) = 0.$$

Hence, $\mu(N) = 0$. Thus, since $X = P \sqcup N$, we have that $\nu \perp \mu$.

(2) $(a) \implies (b)$: Assume $\nu \ll \mu$. We have then that if $\mu(E) = 0$, then $\nu(F) = \mu(F) = 0$ for all F with $F \subset E$, by the monotonicity of μ. Writing

$$X = P \sqcup N, \quad \nu_+(N) = \nu_-(P) = 0,$$

and we have

$$\nu_+(E) = \nu(E \cap F) = 0,$$
$$\nu_-(E) = \nu(E \cap N) = 0,$$

so $|\nu|(E) = \nu_+(E) + \nu_-(E) = 0$. Hence, $|\nu| \ll \mu$.

$(b) \implies (c)$: Assume $|\nu| \ll \mu$. Take $E \in M$ such that $\mu(E) = 0$, then we have $|\nu|(E) = \nu_+(E) + \nu_-(E) = 0$. Since $|\nu|$ is a positive measure, and $\nu_+, \nu_- \le |\nu|$, we get that $\nu_+ \ll \mu$ and $\nu_- \ll \mu$ as well.

$(c) \implies (a)$: Take $E \in M$ with $\mu(E) = 0$. We have that

$$\nu(E) = \nu_+(E) - \nu_-(E) = 0 - 0 = 0.$$

Hence, $\nu \ll \mu$.

□

Problem 66. Let ν be a signed measure on $(X, \mathcal{M})$. Prove the following assertions:

(1) $\mathcal{L}^1(\nu) = \mathcal{L}^1(|\nu|)$.

(2) If $f \in \mathcal{L}^1(\nu)$,

$$\left| \int f d\nu \right| \le \int |f| d|\nu|.$$

(3) If $E \in \mathcal{M}$,

$$|\nu|(E) = \sup \left\{ \left| \int_E f d\nu \right| \ : \ -1 \le f \le 1 \right\}.$$

Proof. (1) Take $f \in \mathcal{L}^1(\nu)$. We wish to show that $f \in \mathcal{L}^1(|\nu|)$. Again, write $\nu = \nu_+ - \nu_-$, and write

$$X = P \sqcup N, \quad \nu_+(N) = \nu_-(P) = 0.$$

We have

$$\int |f| d\nu_+ = \int_P |f| d\nu < \infty,$$
$$\int |f| d\nu_- = -\int_N |f| d\nu > -\infty,$$

so

$$\int |f| d|\nu| = \int |f| (d\nu_+ + d\nu_-) = \int |f| d\nu_+ + \int |f| d\nu_- < \infty.$$

Thus, $f \in L^1(|\nu|)$. The other direction is clear, since if $f \in L^1(|\nu|)$, we have

$$\int |f| d|\nu| = \int |f|(d\nu_+ + d\nu_-) = \int |f| d\nu_+ + \int |f| d\nu_- < \infty,$$

so

$$\int |f| d\nu_+, \int |f| d\nu_- < \infty,$$

and hence $f \in L^1(\nu_+) \cap L^1(\nu_-) = L^1(\nu)$.

(2) We write out the definition;

$$\left| \int f d\nu \right| = \left| \int f d\nu_+ - \int f d\nu_- \right| \leq \left| \int f d\nu_+ \right| + \left| \int f d\nu_- \right|$$
$$\leq \int |f| d\nu_+ + \int |f| d\nu_- = \int |f| d|\nu|.$$

(3) Let

$$K(E) := \sup \left\{ \left| \int_E f d\nu \right| \ : \ -1 \leq f \leq 1 \right\}.$$

Take f measurable so that $-1 \leq f \leq 1$. Then we have

$$\left| \int_E f d\nu \right| \leq \int_E |f| d|\nu| \leq \int_E d|\nu| = |\nu|(E).$$

So $K(E) \leq |\nu|(E)$. For the other direction, write

$$X = P \sqcup N, \quad \nu_+(N) = \nu_-(P) = 0.$$

We have

$$|\nu|(E) = \int_E d|\nu| = \int_E d\nu_+ + \int_E d\nu_- = \int_E \chi_P d\nu - \int_E \chi_N d\nu = \int_E (\chi_P - \chi_N) d\nu \leq \left| \int_E (\chi_P - \chi_N) d\nu \right|,$$

and since

$$-1 \leq \chi_P - \chi_N \leq 1$$

by disjointedness, we have that

$$\chi_P - \chi_N \in \left\{ \left| \int_E f d\nu \right| \ : \ -1 \leq f \leq 1 \right\},$$

so

$$|\nu|(E) \leq K(E).$$

Hence, $|\nu|(E) = K(E)$, as desired.

□

Problem 67. Suppose

$$\nu(E) := \int_E f d\mu, \quad E \in \mathcal{M}$$

where μ is a positive measure on $(X, \mathcal{M})$ and f is an extended μ-integrable function. Describe the Hahn decompositions of ν and the positive, negative, and total variations of ν in terms of f and μ.

Proof. First, we wish to describe a Hahn decomposition of ν in terms of f and μ. That is, we need to find sets P and N so that

$$X = P \sqcup N,$$

and where P is ν positive and N is ν negative. Write

$$P = \{f \geq 0\},$$
$$N = \{f < 0\}.$$

Then clearly $P \cap N = \varnothing$, $P \sqcup N = X$. Let $E \subset P$, then we have

$$\nu(E) = \int_E f d\mu = \int f \cdot \chi_E d\mu \geq 0.$$

So we see that ν is positive on P. Likewise, $E \subset N$ implies that

$$\nu(E) = \int_E f d\mu = \int f \cdot \chi_E d\mu \leq 0,$$

so that ν is negative on N. Hence, we have a Hahn decomposition of ν.

With this, we can find the positive and negative variations of ν. Write

$$\nu_+(E) = \nu(E \cap P) = \int_E f \cdot \chi_P d\mu,$$

$$\nu_-(E) = -\nu(E \cap N) = -\int_E f \cdot \chi_N d\mu.$$

Thus, we write $\nu = \nu_+ - \nu_-$. The total variation then is

$$|\nu| = \nu_+ + \nu_-,$$

which we write as

$$\begin{aligned} |\nu|(E) &= \int_E f \cdot \chi_P d\mu - \int_E f \cdot \chi_N d\mu \\ &= \int_E f(\chi_P - \chi_N) d\mu. \end{aligned}$$

□

Problem 68. Suppose μ is a positive measure on $(X, \mathcal{M})$. Suppose $\{\nu_j\}$ is a sequence of positive measures on $(X, \mathcal{M})$. Prove the following assertions:

(1) If $\{\nu_j\}$ is a sequence of positive measures on $(X, \mathcal{M})$ with $\nu_j \perp \mu$ for all j, then $\sum \nu_j \perp \mu$.
(2) If ν_1, ν_2 are positive measures on $(X, \mathcal{M})$ with at least one of ν_1, ν_2 is finite and $\nu_j \perp \mu$ for $j = 1, 2$, then $(\nu_1 - \nu_2) \perp \mu$.
(3) If $\{\nu_j\}$ is a sequence of positive measures on $(X, \mathcal{M})$ with $\nu_j \ll \mu$ for all j, then $\sum \nu_j \ll \mu$.
(4) If ν_1, ν_2 are positive measures on $(X, \mathcal{M})$ with at least one of ν_1, ν_2 is finite and $\nu_j \ll \mu$ for $j = 1, 2$, then $(\nu_1 - \nu_2) \ll \mu$.

Proof. (1) For each j, we have that

$$X = E_j \sqcup F_j,$$

with μ null on E_j and ν_j null on F_j. Let $E = \bigcup_{j=1}^\infty E_j$, $F = \bigcap_{j=1}^\infty F_j$. Notice first that $X = E \sqcup F$, since these are compliments and DeMorgans applies. Letting $G \subset E$, we can write $G_j = G \cap E_j$, and we have that

$$\mu(G) = \mu\left(\bigcup_j G_j\right) \leq \sum_j \mu(G_j) = 0.$$

Hence, $\mu(G) = 0$, and since this applies for all subsets we have that E is μ null. Letting $G \subset F$, we see that

$$\sum_j \nu_j(G) = \sum_j 0 = 0,$$

since $\nu_j(G) \leq \nu_j(E_j) = 0$ for each j. Hence, we have that $\sum_j \nu_j$ is null on F, so we have that $\sum \nu_j \perp \mu$.

(2) We have
$$X = E_1 \sqcup F_1 = E_2 \sqcup F_2,$$
where μ is null on E_1, E_2 and ν_j is null on F_1, F_2. Let $E = E_1 \cup E_2$, $F = F_1 \cap F_2$. From above, we see that
$$X = E \sqcup F,$$
and μ is null on E. Furthermore, we see that $\nu_1 - \nu_2$ is null on F by the same argument above, so we have that $(\nu_1 - \nu_2) \perp \mu$.

(3) Let $E \in \mathcal{M}$ be such that $\mu(E) = 0$, then we have for all j $\nu_j(E) = 0$, so $\sum \nu_j(E) = 0$. Hence, $\sum \nu_j \ll \mu$.

(4) The same idea as above: take $E \in \mathcal{M}$ such that $\mu(E) = 0$, then $\nu_1(E) = 0$, $\nu_2(E) = 0$, so $(\nu_1 - \nu_2)(E) = 0$. Since the choice of E was arbitrary, we have $(\nu_1 - \nu_2) \ll \mu$.

□

Problem 69. Let $F : \mathbb{R} \to \mathbb{R}$ be a non-decreasing continuously differentiable function, and let μ_F be the corresponding Lebesgue-Stieltjes measure on $\mathbb{R}$. Prove that $\mu_F \ll \lambda$ (Lebesgue measure) and
$$\frac{d\mu_F}{d\lambda} = F' \ \lambda\text{-a.e.}$$
In other words, prove that
$$\mu_F(E) = \int_E F' d\lambda \ \ \forall E \in \mathcal{B}_{\mathbb{R}}.$$

Proof. We can define a measure ν such that
$$\nu(E) := \int_E F' d\lambda$$
or in other words, $d\nu = F' d\lambda$. Write the real line as a countable disjoint union of half open intervals $(a, b]$. Observe that
$$\nu((a, b]) = \int_{(a,b]} F' d\lambda = F(b) - F(a) = \mu_F((a, b]) < \infty.$$
Let Π be the collection of half open intervals on $\mathbb{R}$, along with the empty set. By what we've just shown, we see that $\mu = \nu$ on Π. Notice that, by argument laid out in **Problem 12**, we get that this is actually a Π system (simply go through the cases and see that intersecting two half open intervals gives either a half open interval or the empty set, use this and induction to get that it holds for a finite number of half open intervals). Hence, using **Problem 6**, we get that $\mu = \nu$ on $\mathcal{B}_{\mathbb{R}}$. So we have $d\mu = F' d\lambda$, This gives us that $F' = \frac{d\mu_F}{d\lambda}$ λ-a.e. Next, notice that if $\lambda(E) = 0$, we have that
$$\mu_F(E) = \int_E F' d\lambda \leq \lambda(F) \cdot ||F'||_\infty = 0,$$
so $\mu_F \ll \lambda$. □

Problem 70. If $f_n \to f$ in measure, and if there is a function ϕ such that $f_n \le \phi$ a.e. for all $n \ge 1$, prove that $f \le \phi$ a.e. as well.

Proof. By definition, $f_n \to f$ in measure implies that for all $\epsilon > 0$,

$$\mu(\{x \ : \ |f_n(x) - f(x)| \ge \epsilon\}) \to 0.$$

Examine the set

$$E_\epsilon := \{x \ : \ f - \phi \ge \epsilon\}.$$

Then we can write this as

$$E_\epsilon = \{x \ : \ f - f_n + f_n - \phi \ge \epsilon\} \subset \{x \ : \ f - f_n \ge \epsilon/2\} \cup \{x \ : \ f_n - \phi \ge \epsilon/2\}.$$

Notice that we can write

$$\{x \ : \ f_n(x) - f(x) \ge \epsilon\} \subset \{x \ : \ |f_n(x) - f(x)| \ge \epsilon\},$$

and so we see that we can make the measure arbitrarily small. Hence, take it so that the measure is 0.

Since $f_n \le \phi$ a.e. for all $n \ge 1$, we have that $0 \le \phi - f_n$ a.e., or in other words the set where $f_n - \phi \le \epsilon$ has measure 0 for all $\epsilon > 0$.

So, we have that $E_\epsilon = 0$. Notice now that $E_\epsilon \nearrow E := \{x \ : \ f - \phi \ge 0\} = \{x \ : \ f \ge \phi\}$, and so $\mu(E) = 0$ by continuity from below. □

Problem 71. Show that convergence in L^1 implies convergence in measure.

Proof. Convergence in L^1 says

$$\int |f - f_n| \to 0.$$

Notice that we have

$$\int |f - f_n| = \int_E |f - f_n| + \int_{E^c} |f - f_n|,$$

where $E = \{x \ : \ |f(x) - f_n(x)| \ge \epsilon\}$ for some $\epsilon > 0$. Hence,

$$\int_E |f - f_n| \le \int |f - f_n|.$$

Furthermore, since $|f - f_n| \ge \epsilon$ on E, we have

$$\int_E |f - f_n| \ge \int_E \epsilon = \epsilon\mu(E).$$

So we have

$$\mu(E) \le \frac{1}{\epsilon} \int |f - f_n|.$$

The right hand side goes to 0, and so we see that the left hand side goes to 0. Since this applies for all $\epsilon > 0$, we win. □

Problem 72. Suppose $A \subset B$ are Lebesgue measurable subsets and $\lambda(A) = \lambda(B)$. Then show that any set C such that $A \subset C \subset B$ is also measurable and that $\lambda(C) = \lambda(A)$.

Proof. Notice that

$$\lambda(A) \le \lambda(C) \le \lambda(B) = \lambda(A) \implies \lambda(C) = \lambda(A).$$

Notice that we have $\lambda(B - A) = 0$. Furthermore, we get that $C - A \subset B - A$, and so $C - A$ is measurable by completeness. Hence, $C = A \cup (C - A)$ is a union of measurable sets, and so measurable. □

Problem 73 (Axler 2.1)**.** Suppose that $A, B \subset \mathbb{R}$, $\lambda(B) = 0$, then $\lambda(A \cup B) = \lambda(A)$ (λ here is the Lebesgue measure).

Proof. Montonicity says
$$A \subset A \cup B \implies \lambda(A) \leq \lambda(A \cup B).$$
Subadditivity gives
$$\lambda(A \cup B) \leq \lambda(A) + \lambda(B) = \lambda(A).$$
Hence, we have
$$\lambda(A) \leq \lambda(A \cup B) \leq \lambda(A) \implies \lambda(A \cup B) = \lambda(A).$$
□

Problem 74 (Axler 2.3)**.** Prove that if $A, B \subset \mathbb{R}$, $\lambda(A) < \infty$, then $\lambda(B - A) \geq \lambda(B) - \lambda(A)$.

Proof. Notice that we have
$$B = A \sqcup (B - A),$$
so we have
$$\lambda(B) \leq \lambda(A) + \lambda(B - A).$$
Since $\lambda(A) < \infty$, we can subtract from both sides to get
$$\lambda(B) - \lambda(A) \leq \lambda(B - A).$$
□

Problem 75 (Axler 2.17)**.** Suppose X is a Borel subset of $\mathbb{R}$, and $f : X \to \mathbb{R}$ is a function such that
$$F := \{x \in X \ : \ f \text{ is not continuous at } x\}$$
is a countable set. Prove f is a Borel measurable function.

Proof. Let $E \subset \mathbb{R}$ open. Then we have that $f^{-1}(E) = (f^{-1}(E) \cap F) \cup (f^{-1}(E) \cap F^c)$. Since λ is a complete measure, we have that $f^{-1}(E) \cap F$ is measurable, since it has measure 0, and since f is continuous on F^c, we get that $f^{-1}(E) \cap F^c$ is measurable. Hence, $f^{-1}(E)$ is measurable. Since this works for all $E \subset \mathbb{R}$ open, we have that f is measurable. □

Problem 76. Let $\mathcal{M}$ be an algebra. If $\mathcal{M}$ is closed under countable disjoint unions, then it is a σ-algebra.

Proof. Recall that an algebra is a collection of sets which is closed under complements and finite unions and $\mathcal{M} \neq \varnothing$. We also give it the property that it is closed under countable disjoint unions; that is, if $\{E_i\} \subseteq \mathcal{M}$ is a collection of disjoint sets (that is, $E_i \cap E_j = \varnothing$ if $i = j$), then $\bigsqcup E_i \in \mathcal{M}$.

We want to show that $\mathcal{M}$ is a σ-algebra. Recall that a σ-algebra is a collection of sets Y such that

- $X \in Y$,
- Y is closed under complements,
- Y is closed under countable unions.

So we have that $\mathcal{M}$ satisfies the first two properties, and so it suffices to show it satisfies the last. Notice that DeMorgans gives us $(\bigcup_{i=1}^n E_i^C)^C = \bigcap_{i=1}^n E_i$, and so $\mathcal{M}$ is also closed under finite intersections.

Let $\{E_i\}$ be a collection of sets in $\mathcal{M}$. Let $F_1 = E_1$, $F_n = E_n \cap \left(\bigcup_{i=1}^{n-1} E_i\right)^C$. Since the $E_i \in \mathcal{M}$, we get that $F_i \in \mathcal{M}$ for all i. Moreover, $F_i \cap F_j = \varnothing$; they are disjoint. So by assumption, $\bigsqcup F_i \in \mathcal{M}$. But notice as well that $\bigsqcup F_i = \bigcup E_i$, and hence $\bigcup E_i \in \mathcal{M}$. So $\mathcal{M}$ is closed under countable unions. □

Problem 77 (Folland 1.5)**.** If $\mathcal{M}$ is a σ-algebra generated by E, then $\mathcal{M}$ is the union of the σ-algebras generated by F as F ranges over all countable subsets of E.

Proof. We proceed via the hint. That is, we first wish to show that

$$G := \bigcup_{\substack{F \subset \mathcal{P}(E) \\ F \text{ countable}}} \sigma(F)$$

is a σ-algebra. First, notice that $\varnothing \in G$ clearly. Next, let $A \in G$. Then we have that $A \in \sigma(F)$ for some F, and so $A^c \in \sigma(F)$, which implies it's in the union. Finally, let $\{A_i\}$ be a collection of subsets of G. We wish to then show that $\bigcup A_i \in G$. Notice that $A_i \in F_i$ for some i, where we may have $F_i = F_j$ for $i \neq j$. Since this is a countable union, we have that we can take a union over these F_i to have a countable union; that is,

$$\bigcup A_i \in \sigma\left(\bigcup F_i\right) \subset G.$$

So G is closed under countable unions, and so is a σ-algebra.

Now, we see that $\mathcal{M} \subset G$, since G is a σ-algebra which contains E. For the other direction, just note that $\sigma(F) \subset \mathcal{M}$ for all F, and so $\bigcup \sigma(F) = G \subset \mathcal{M}$. Hence, $\mathcal{M} = G$. □

Problem 78 (Folland 1.6)**.** Complete the proof of **Theorem 1.9**. That is, the following:

Suppose that $(X, \mathcal{M}, \mu)$ is a measure space. Let $\mathcal{N} = \{N \in \mathcal{M} \ : \ \mu(N) = 0\}$, and $\overline{\mathcal{M}} = \{E \cup F \ : E \in \mathcal{M} \text{ and } F \subset N \text{ for some } N \in \mathcal{N}\}$. Then $\overline{\mathcal{M}}$ is a σ-algebra, and there is a unique extension $\overline{\mu}$ of μ to a complete measure on $\overline{\mathcal{M}}$.

Proof. Since $\mathcal{M}$ and $\mathcal{N}$ are closed under countable unions, we clearly get $\overline{\mathcal{M}}$ is as well. If $E \cup F \in \overline{\mathcal{M}}$ where $E \in \mathcal{M}$ and $F \subset N \in \mathcal{N}$, we can assume that $E \cap N = \varnothing$, otherwise disjointify them. Then $E \cup F = (E \cup N) \cap (N^C \cup F)$, so $(E \cup F)^c \in \overline{\mathcal{M}}$. Hence, its a σ-algebra.

We then want to show that $\nu = \overline{\mu}$ is a complete measure on $\overline{\mathcal{M}}$. To check that it's a measure, we need to show two things:

(1) $\nu(\varnothing) = 0$: This follows, since $\varnothing = \varnothing \cup \varnothing$, and so $\nu(\varnothing) = \mu(\varnothing) = 0$.
(2) If E_i is a disjoint collection of sets in $\overline{\mathcal{M}}$, we have that $E_i = A_i \cup F_i$, where $A_i \in \mathcal{M}$ and $F_i \subset N_i$ for some $N_i \in \mathcal{N}$. Hence, we have

$$\bigsqcup E_i = \bigsqcup (A_i \cup F_i) = \left(\bigsqcup A_i\right) \cup \left(\bigsqcup F_i\right),$$

and so

$$\nu\left(\bigsqcup E_i\right) = \mu\left(\bigsqcup A_i\right) = \sum \mu(A_i) = \sum \nu(E_i).$$

So ν is a measure. Moreover, we see ν is a complete measure; we see that if $N \subset X$ is such that $N \subset F$, $\mu(F) = 0$, then we have that

$$N = \varnothing \cup N \in \overline{\mathcal{M}}, \nu(N) = 0.$$

So it is a complete measure. Finally, we check the uniqueness of ν. Assume that we have another measure, γ, which is equal to μ on $\mathcal{M}$. Given $A \in \overline{\mathcal{M}}$, we have that $A = E \cup F$, where $E \in \mathcal{M}$, $F \subset N \in \mathcal{M}$, $\mu(N) = 0$. Hence, we have that $E \subset A \subset E \cup N$, and so we get

$$\gamma(E) = \mu(E) \leq \gamma(A) \leq \gamma(E \cup N) = \mu(E \cup N) = \mu(E) = \gamma(E),$$

and so we have $\gamma(A) = \gamma(E) = \nu(E)$. Hence, the measure is unique. □

Problem 79 (Folland 1.7)**.** If $\mu_1, \ldots, \mu_n$ are measures on $(X, \mathcal{M})$, and $a_1, \ldots, a_n \in [0, \infty)$, then $\sum a_i \mu_i$ is a measure.

Proof. We need to check the two assumptions. Notice that

$$\sum a_i \mu_i(\varnothing) = \sum a_i(0) = 0,$$

and so the first assumption holds. Next, take $\{E_j\} \subset \mathcal{M}$ disjoint. Then we have that

$$\mu_i \left(\bigsqcup E_j \right) = \sum \mu_i(E_j),$$

and so

$$\sum a_i \mu_i \left(\bigsqcup E_j \right) = \sum_i a_i \left(\sum_j \mu_i(E_j) \right) = \sum_j \sum_i a_i \mu(E_j).$$

Hence, it's a measure. □

Problem 80 (Folland 1.8)**.** If $(X, \mathcal{M}, \mu)$ is a measure space, and $\{E_j\} \subset \mathcal{M}$, then

$$\mu\left(\liminf E_j\right) \leq \liminf \mu(E_j).$$

Also,

$$\mu(\limsup E_j) \geq \limsup \mu(E_j),$$

provided that

$$\mu\left(\bigcup E_j\right) < \infty.$$

Proof.

(1) Recall that

$$\liminf E_j = \bigcup_{j=1}^{\infty} \bigcap_{m=j}^{\infty} E_m.$$

By subadditivity, we see that

$$\mu\left(\bigcap_{m=j}^{\infty} E_m \right) \leq \mu(E_j)$$

for all $j \geq m$, and so in particular we have

$$\mu\left(\bigcap_{m=j}^{\infty} E_m \right) \leq \inf_{m \geq j} \mu(E_m).$$

Taking the limit as $j \to \infty$ of both sides gives

$$\mu(\liminf E_j) \leq \liminf \mu(E_j).$$

(2) This is proven analogously. Notice that

$$\mu\left(\bigcup_{m=j} E_m \right) \geq \mu(E_j)$$

for all $j \geq m$. Hence, we get

$$\mu\left(\bigcup_{m=j} E_m \right) \geq \sup_{m \geq j} \mu(E_m)$$

Since the union has finite measure, we get the desired result by taking limits.

□

Problem 81 (Folland 1.9)**.** If $(X, \mathcal{M}, \mu)$ is a measure space and $E, F \in \mathcal{M}$, then $\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)$.

Proof. Recall that we can write
$$E = (E \cap F) \sqcup (E \cap F^C).$$
Hence,
$$\mu(E) = \mu(E \cap F) + \mu(E \cap F^C).$$
Likewise, we can write
$$E \cup F = (E \cap F^c) \sqcup F.$$
So we have
$$\mu(E \cup F) + \mu(E \cap F) = \mu(F) + \mu(E \cap F^c) + \mu(E \cap F) = \mu(F) + \mu(E).$$
□

Problem 82 (Folland 1.10)**.** Given a measure space $(X, \mathcal{M}, \mu)$ and $E \in \mathcal{M}$, define $\mu_E(A) = \mu(A \cap E)$ for $A \in \mathcal{M}$. Then μ_E is a measure.

Proof. Notice that
$$\mu_E(\varnothing) = \mu(E \cap \varnothing) = \mu(\varnothing) = 0.$$
Notice as well that if $\{E_i\} \subset \mathcal{M}$ is such that it is disjoint, then
$$\mu_E\left(\bigsqcup E_i\right) = \mu\left(\left(\bigsqcup E_i\right) \cap E\right) = \mu\left(\bigsqcup (E_i \cap E)\right) = \sum_i \mu(E_i \cap E) = \sum_i \mu_E(E_i).$$
Hence, it is a measure. □

Problem 83 (Folland 1.11)**.** A finitely additive measure μ is a measure if and only if it satisfies the conclusion of **Theorem 1.8c**. If $\mu(X)$ is finite, μ is a measure if and only if it satisfies the conclusion of **Theorem 1.8d**.

Proof. ($\Longrightarrow$) Clear.
($\Longleftarrow$) It suffices to show that if $E_i \subset \mathcal{M}$ is disjoint, then $\mu\left(\bigcup E_i\right) = \sum \mu(E_i)$. By finite additivity, we get
$$\mu\left(\bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n \mu(E_i).$$
We can then do limits to get
$$\mu\left(\bigcup_{i=1}^\infty E_i\right) = \sum_{i=1}^\infty \mu(E_i).$$
□

Problem 84 (Folland 1.12)**.** Let $(X, \mathcal{M}, \mu)$ be a finite measure space.

1. (1) If $E, F \in \mathcal{M}$ and $\mu(E \Delta F) = 0$, then $\mu(E) = \mu(F)$.
2. (2) Say that $E \sim F$ if $\mu(E \Delta F) = 0$; then $\sim$ is an equivalence relation on $\mathcal{M}$.
3. (3) For $E, F \in \mathcal{M}$, define $\rho(E, F) = \mu(E \Delta F)$. Then $\rho(E, G) \leq \rho(E, F) + \rho(F, G)$, and hence ρ defines a metric on the space $\mathcal{M}/\sim$ of equivalence classes.

Proof.

(1) Recall that
$$E\Delta F = (E - F) \sqcup (F - E).$$
Hence, we have
$$\mu(E\Delta F) = \mu(E - F) + \mu(F - E) = 0.$$
This implies that $\mu(E - F) = \mu(F - E) = 0$. But notice that $E \subset F \sqcup (E - F)$, and so we get
$$\mu(E) \leq \mu(F) + \mu(E - F) = \mu(F).$$
Likewise, $F \subset E \sqcup (F - E)$, and so $\mu(F) \leq \mu(E)$. Thus, $\mu(E) = \mu(F)$.

(2) We first see that $E\Delta E = \varnothing$, and so $E \sim E$. Likewise, if $E \sim F$, then $\mu(E\Delta F) = \mu(E - F) + \mu(F - E) = \mu(F - E) + \mu(E - F) = \mu(F\Delta E) = 0$, and so $F \sim E$. Finally, if $E \sim F$, $F \sim G$, then we have that
$$\mu(E\Delta G) = \mu(E - G) + \mu(G - E).$$
Notice that $E - G \subset (E - F) \cup (F - G)$, and so $\mu(E - G) = 0$. An analogous argument applies for $G - E$.

(3) By the observation prior, we have
$$\mu(E\Delta G) = \mu(E-G)+\mu(G-E) \leq \mu(E-F)+\mu(F-G)+\mu(G-F)+\mu(F-E) = \mu(E\Delta F)+\mu(G\Delta F).$$
Hence, we get the triangle inequality. On the quotient space, it's clear that this will be a metric, then.

□

Problem 85 (Folland 1.13)**.** Every σ-finite measure is semifinite.

Proof. Since μ is σ-finite, we can write
$$X = \bigcup F_i,$$
where $\mu(F_i) < \infty$. Take $E \subset X$ such that $\mu(E) = \infty$. Then we have that
$$E = \bigcup (F_i \cap E),$$
and so
$$\infty = \mu(E) \leq \sum_i \mu(F_i \cap E),$$
and hence $\sum_i \mu(F_i \cap E) = \infty$. So there must be at least one N such that $\mu(F_N \cap E) > 0$. Furthermore, we have $\mu(F_N \cap E) \leq \mu(F_N) < \infty$. So, we have
$$F_N \cap E \subset F,\ 0 < \mu(F_N \cap E) < \infty.$$
The choice of E was arbitrary, and so our measure is semifinite. □

Problem 86 (Folland 1.17)**.** If μ^* is an outer measure on X and $\{A_j\}$ is a sequence of disjoint measurable sets, then
$$\mu^*\left(E \cap \bigcup A_j\right) = \sum \mu^*(E \cap A_j).$$

Proof. Notice immediately that we have
$$\mu^*\left(E \cap \bigcup A_j\right) = \mu^*\left(\bigcup (E \cap A_j)\right) \leq \sum \mu^*(E \cap A_j),$$
by properties of outer measure. We then need to show the other direction.

Let $B_n = \bigcup_1^n A_j$, $B = \bigcup_{n=1}^\infty B_n$. Since A_j measurable, we have
$$\begin{aligned}\mu^*(E \cap B_n) &= \mu^*(E \cap B_n \cap A_n) + \mu^*(E \cap B_n \cap A_n^c)\\ &= \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1}).\end{aligned}$$

By induction we have

$$\mu^*(E \cap B_n) = \sum_1^n \mu^*(E \cap A_j).$$

Monotonicity tells us that

$$\mu^*(E \cap B) \geq \sum_1^n \mu^*(E \cap A_j).$$

Since this applies for all n, we have

$$\mu^* \left(E \cap \bigcup A_j \right) \geq \sum \mu^*(E \cap A_j),$$

as desired. □

Problem 87 (Folland 1.18)**.** Let $\mathcal{A}$ be an algebra, $\mathcal{A}_\sigma$ the collection of countable unions of things in $\mathcal{A}$, and $\mathcal{A}_{\sigma\delta}$ the collection of countable intersections of sets in $\mathcal{A}_\sigma$. Let μ_0 be a premeasure and μ^* be the induced outer measure.

(1) For any $E \subset X$ and $\epsilon > 0$, there exists $A \in \mathcal{A}_\sigma$ with $E \subset A$ and $\mu^*(A) \leq \mu^*(E) + \epsilon$.
(2) If $\mu^*(E) < \infty$, then E is μ^* measurable if and only if there exists $B \in A_{\sigma\delta}$ with $E \subset B$ and $\mu^*(B - E) = 0$.
(3) We could take μ_0 to be σ-finite instead of $\mu^*(E) < \infty$.

Proof.

(1) This follows by definition of the induced outer measure; we have

$$\mu^*(E) = \inf \left\{ \sum \mu_0(A_j) \ : \ A_j \in \mathcal{A}, E \subset \bigcup A_j \right\}.$$

Hence, for all $\epsilon > 0$, there is a $B \in A_\sigma$ such that

$$\mu^*(B) \leq \mu^*(E) + \epsilon.$$

(2) ($\implies$) Assume that E is μ^* measurable and has finite measure. From above, we can find B such that

$$\mu^*(B) \leq \mu^*(E) + \epsilon.$$

Let B_n be such that

$$\mu^*(B_n) \leq \mu^*(E) + 1/n.$$

We have that

$$\mu^*(B_n) = \mu^*(B_n \cap E) + \mu^*(B_n \cap E^c),$$

and since $E \subset B_n$, we get

$$\mu^*(B_n) = \mu^*(E) + \mu^*(B_n - E).$$

Since $\mu^*(E)$ is finite, this means

$$\mu^*(B_n) - \mu^*(E) = \mu^*(B_n - E).$$

Furthermore, we have

$$\mu^*(B_n) - \mu^*(E) = \mu^*(B_n - E) \leq 1/n.$$

Let $B = \bigcap B_n$. Then

$$\mu^*(B - E) \leq \mu^*(B_n - E) \leq 1/n$$

for all n, and so we have

$$\mu^*(B - E) = 0.$$

Furthermore, since $E \subset B_n$ for all n, we have $E \subset B$, as desired.

($\Longleftarrow$) Assume we have that there is a B such that $E \subset B$ and $\mu^*(B - E) = 0$. Hence, we see that $\mu^*(B) = \mu^*(E)$. Let F be any subset of X.

$$\mu^*(F\cap E)+\mu^*(F\cap E^c) \le \mu^*(F\cap B)+\mu^*(F\cap E^c) \le \mu^*(F\cap B)+\mu^*(F\cap B^c)+\mu^*(B\cap E^c) = \mu^*(F).$$

Since this applies for all F, we have that E is measurable.

(3) Let $X = \bigcup_n X_n$, $\mu^*(X_n) < \infty$. Take $E \subset X$, then $E = \bigcup_n (E \cap X_n) = \bigcup E_n$. Notice that $\mu^*(E_n) < \infty$ for each n. For each n, choose A_{nj} such that $\mu^*(A_{nj} - E_n) \le 1/j2^{-n}$. Then $B_j = \bigcup_n A_{nj}$ is such that $\mu^*(B_j - E) < 1/j$. Hence, taking $B = \bigcap_j B_j$, we have $\mu^*(B - E) = 0$. The rest follows.

□

Problem 88 (Folland 1.19)**.** Let μ^* be an outer measure on X induced from a finite premeasure μ_0. If $E \subset X$, define the inner measure of E to be $\mu_*(E) = \mu_0(X) - \mu^*(E^c)$. Then E is μ^* measurable if and only if $\mu^*(E) = \mu_*(E)$.

Proof. ($\Longrightarrow$) Assume that E is μ^* measurable. Then we have that

$$\mu^*(X) = \mu^*(E) + \mu^*(E^c).$$

Since X is μ_0 measurable, we have that $\mu_0(X) < \infty$. Hence, $\mu^*(E) + \mu^*(E^c) < \infty$, and so each is finite. We then have that we can subtract things, and so

$$\mu_0(X) = \mu^*(E) + \mu^*(E^c) \iff \mu_*(E) = \mu^*(E).$$

($\Longleftarrow$) Assume that $\mu^*(E) = \mu_0(X) - \mu^*(E^c)$. We have that there are $E \subset A_n$ such that

$$\mu^*(A_n) \le \mu^*(E) + 1/n.$$

Notice that

$$\mu^*(E^c) = \mu^*(A_n^c) + \mu^*(A_n - E).$$

Notice as well that

$$\mu^*(X) = \mu^*(A_n) + \mu^*(A_n^c),$$

and so we have

$$\mu^*(E) = \mu_0(X) - \mu^*(E^c) = \mu^*(A_n) + \mu^*(A_n^c) - \mu^*(A_n^c) - \mu^*(A_n - E).$$

Hence,

$$\mu^*(A_n) - \mu^*(E) = \mu^*(A_n - E).$$

Taking $A = \bigcap A_n$, we have that

$$\mu^*(A - E) = 0,$$

and so E is μ^* measurable. □

Problem 89 (Folland 1.24)**.** Let μ be a finite measure on $(X, \mathcal{M})$, and let μ^* be the outer measure induced by μ. Suppose that $E \subset X$ satisfies $\mu^*(E) = \mu^*(X)$.

(1) If $A, B \in \mathcal{M}$ and $A \cap E = B \cap E$, then $\mu(A) = \mu(B)$.

(2) Let $\mathcal{M}_E = \{A \cap E \ : \ A \in \mathcal{M}\}$, and define the function ν on $\mathcal{M}_E$ defined by $\nu(A \cap E) = \mu(A)$. Then $\mathcal{M}_E$ is a σ-algebra on E and ν is a measure on $\mathcal{M}_E$.

Proof.

(1) Recall from **Folland 1.12**, $\mu(E\Delta F) = 0$ implies $\mu(E) = \mu(F)$. So it suffices to show that $\mu(B - A) = 0$ and $\mu(A - B) = 0$.

Notice that $(A - B) \cap E = A \cap B^c \cap E = (A \cap E) \cap B^c = (B \cap E) \cap B^c = \varnothing$, so $A - B \subset E^c$, or $E \subset (A - B)^c$. This implies that $\mu^*(E) = \mu^*(X) \leq \mu^*((A - B)^c) \leq \mu^*(X)$, and so $\mu^*((A - B)^c) = \mu^*(X)$. Since $(A - B) \in \mathcal{M}$, this implies that

$$\mu^*((A - B)^c) + \mu^*(A - B) = \mu^*(X),$$

and since it's a finite measure we get $\mu^*(A - B) = 0$. An analogous argument gives us $\mu^*(B - A) = 0$, so we have $\mu(A) = \mu(B)$.

(2) We need to check three things for a σ-algebra. First, it's clear to see that $\varnothing \in \mathcal{M}_E$, so it's nonempty. Second, we check it's closed under complements. That is, if $A \cap E \in \mathcal{M}_E$, then $(A \cap E)^c \cap E \in \mathcal{M}_E$. Use DeMorgans to write this as

$$(A \cap E)^c = A^c \cup E^c,$$

and so

$$(A \cap E)^c \cap E = (A^c \cup E^c) \cap E = A^c \cap E \in \mathcal{M}_E.$$

Finally, we need to check it's closed under countable unions. This, however, follows directly from the fact that $\mathcal{M}$ is a σ-algebra.

Next, we need to check that ν is a measure. First, we see that $\nu(\varnothing) = \nu(\varnothing \cap E) = \mu(\varnothing) = 0$. Second, let $F_i \cap E$ be a disjoint collection in $\mathcal{M}_E$. Then

$$\nu\left(\bigsqcup F_i \cap E\right) = \nu\left(\left(\bigsqcup F_i\right) \cap E\right) = \mu\left(\bigsqcup F_i\right) = \sum \mu(F_i).$$

□

Problem 90 (Folland 1.25)**.** Complete the proof of **Theorem 1.19**. That is, suppose $E \in \mathcal{M}_\mu$ is arbitrary. Show that this implies that $E = V - N$, where V is a G_δ set and $\mu(N) = 0$ (this is sufficient for establishing (c)).

Proof. We have it for the case $\mu(E) < \infty$. Since we are working over $\mathbb{R}$, we have that it is σ-finite. Let $E_n = [-n, n] \cap E$. We can find U_n open such that $\mu(U_n) \leq \mu(E_n) + \epsilon/2^n$, and so taking $U = \bigcup U_n$, we have

$$\mu(U - E) = \mu\left(\bigcup_n (U_n - E_n)\right) \leq \sum \mu(U_n - E_N) < \epsilon.$$

Take V_n to be U such that $\epsilon = 1/n$. Intersecting over all V_n gives

$$\mu(V - E) = 0.$$

Hence,

$$E = V - (V - E).$$

□

Problem 91 (Folland 1.26)**.** If $E \in \mathcal{M}_\mu$ and $\mu(E) < \infty$, then for every $\epsilon > 0$ there is a set A that is a finite union of open intervals such that $\mu(E\Delta A) < \epsilon$.

Proof. We have that there are compact K and open U such that $K \subset E \subset U$, and $\mu(U) \leq \mu(E) + \epsilon/2$, $\mu(E) \leq \mu(K) + \epsilon/2$. We can write $U = \bigsqcup_n I_n$, where I_n are disjoint open intervals. Since K compact, and these cover K, we have that we can get a finite subcover $\bigcup_1^N I_n$ which cover K. Furthermore, calling this A, we have $K \subset A \subset U$. So, we get

$$\mu(E\Delta A) = \mu(E - A) + \mu(A - E) \leq \mu(E - K) + \mu(U - E) < \epsilon.$$

□

Problem 92 (Folland 1.29)**.** Let E be a Lebesgue measurable set.

1. (1) If $E \subset N$, where N is the nonmeasurable set described in 1.1 (the Vitali set), then $m(E) = 0$.
2. (2) If $m(E) > 0$, then E contains a nonmeasurable set.

Proof.

1. (1) We have that
$$\bigcup_{r \in R} E_r \subset \bigcup_{r \in R} N_r = [0, 1).$$
So
$$\sum_{r \in R} m(E) \leq 1.$$
Since R is infinite, this forces $m(E) = 0$.
2. (2) Omitted (tedious but doable) TODO

□

Problem 93 (Folland 2.1)**.** Let $f : X \to \overline{\mathbb{R}}$ and $Y = f^{-1}(\mathbb{R})$. Then f measurable if and only if $f^{-1}(\{\infty\}) \in \mathcal{M}$, $f^{-1}(\infty) \in \mathcal{M}$, and f measurable on Y.

Proof. ($\Longrightarrow$) If f measurable, we have $f^{-1}(\pm\infty) \in \mathcal{M}$, and furthermore f measurable on Y.
($\Longleftarrow$) Suppose we have the following conditions. Then we need to show that for every $A \in \mathcal{B}(\mathbb{R})$, $f^{-1}(A) \in \mathcal{M}$. Notice we can write $A = (A \cap \{\infty\}^c \cap \{-\infty\}^c) \sqcup (A \cap \{\infty\} \cap \{-\infty\}^c) \sqcup (A \cap \{\infty\}^c \cap \{-\infty\})$. Hence, $f^{-1}\left((A \cap \{\infty\}^c \cap \{-\infty\}^c) \sqcup (A \cap \{\infty\} \cap \{-\infty\}^c) \sqcup (A \cap \{\infty\}^c \cap \{-\infty\})\right) = f^{-1}(A \cap \{\infty\}^c \cap \{-\infty\}^c) \sqcup f^{-1}(A \cap \{\infty\} \cap \{-\infty\}^c) \sqcup f^{-1}(A \cap \{\infty\}^c \cap \{-\infty\})$. Naturally the first is measurable, since it's in Y, and the second is measurable, since it's either the empty set or the point at $\pm\infty$. So we have that it's measurable □

Problem 94 (Folland 2.3)**.** If $\{f_n\}$ a sequence of measurable functions on X, then $\{x : \lim f_n(x) \text{ exists}\}$ is measurable.

Proof. Since $\{f_n\}$ measurable, we have $g = \liminf f_n$ and $h = \limsup f_n$ are both measurable. Furthermore, $k = h - g$ is measurable, and this set is described as $k^{-1}(\{0\})$. □

Problem 95 (Folland 2.4)**.** If $f : X \to \overline{\mathbb{R}}$ and $f^{-1}((r, \infty]) \in \mathcal{M}$ for each $r \in \mathbb{Q}$, then f is measurable.

Proof. By the density of the rationals, we can write
$$(a, \infty] = \bigcup_{r \in X} (r, \infty], \quad X = \{r \in \mathbb{Q} \ : \ r > a\} \subset \mathbb{Q}.$$
Hence, we have
$$f^{-1}((a, \infty]) = \bigcup_{r \in X} f^{-1}((r, \infty]).$$
□

Problem 96 (Folland 2.5)**.** If $X = A \cup B$ where $A, B \in \mathcal{M}$, a function f on X is measurable if and only if f is measurable on A and on B.

Proof. ($\Longrightarrow$) Assume f is measurable, then we clearly have for all E measurable that $f^{-1}(E) \cap A$ is measurable. Same for B.
($\Longleftarrow$) Assume it is measurable on A and B. That is, for all E measurable, we have $f^{-1}(E) \cap A$ is measurable and $f^{-1}(E) \cap B$ is measurable. Since $A \cup B = X$, we have
$$f^{-1}(E) = f^{-1}(E) \cap X = f^{-1}(E) \cap (A \cup B) = (f^{-1}(E) \cap A) \cup (f^{-1}(E) \cap B),$$
and so this is measurable. Hence, f is measurable. □

Problem 97 (Folland 2.6)**.** The supremum of an uncountable family of measurable $\overline{\mathbb{R}}$-valued functions on X can fail to be measurable.

Proof. Let $V \subset [0,1] \subset \mathbb{R}$ be the Vitali set. Take $f_x = \chi_{\{x\}}$ to be the family of functions, where here $x \in V$. Then each f_x is measurable (since it's the characteristic function of a point), but the uncountalbe supremum gives you χ_V, which is non-measurable. □

Problem 98 (Folland 2.10)**.** Prove **Proposition 2.11**. That is, prove the following statement: We have the following if and only if μ is complete:

(1) If f is measurable and $f = g$ μ-a.e., then g is measurable.
(2) If f_n is measurable for $n \in \mathbb{N}$ and $f_n \to f$ μ-a.e., then f is measurable.

Proof. (1) ($\implies$) Assume μ is complete. Let $A = \{f = g\}$, $B = \{f \neq g\}$, then $A \sqcup B = X$. Take $F \in \mathcal{N}$, we wish to show that $g^{-1}(F) \in \mathcal{M}$. Notice we can write

$$g^{-1}(F) = \left(g^{-1}(F) \cap A\right) \sqcup \left(g^{-1}(F) \cap B\right).$$

The set on the left is measurable, since f is measurable, and the set on the right is measurable, since μ is complete. Hence, we have it's measurable.

($\impliedby$) Assume we have the condition. Let $F \subset X$ be such that $\mu(F) = 0$. Let $f = 0$, $g = \chi_F$. Then $g = f$ μ-a.e., so g is measurable, and furthermore $g^{-1}(\{1\}) = F \in \mathcal{M}$. Hence, μ is complete.

(2) ($\implies$) Let $A = \{f_n \to f\}$ and $B = \{f_n \not\to f\}$. Again, $A \sqcup B = X$. The same argument applies here; taking $F \in \mathcal{N}$, we get

$$f^{-1}(F) = \left(f^{-1}(F) \cap A\right) \sqcup \left(f^{-1}(F) \cap B\right)$$

is measurable. (To be more rigorous, let g be the pointwise limit of f_n; we have that g is measurable, and then use (1) to get the desired result).

($\impliedby$) Assume we have the condition. Let $f_n = 0$, $f = \chi_F$, where $F \subset X$ is such that $\mu(F) = 0$. Then $f_n \to f$ μ-a.e., and again we get that $f^{-1}(\{1\}) = F$ is measurable. □

Problem 99 (Folland 2.11)**.** Suppose that f is a function on $\mathbb{R} \times \mathbb{R}^k$ such that f_x is Borel measurable for each $x \in \mathbb{R}$ and f^y is continuous for each $y \in \mathbb{R}^k$. For $n \in \mathbb{N}$, define f_n as follows: for $i \in \mathbb{Z}$, let $a_i = i/n$, and for $a_i \leq x \leq a_{i+1}$ let

$$f_n(x, y) := \frac{f(a_{i+1}, y)(x - a_i) - f(a_i, y)(x - a_{i+1})}{a_{i+1} - a_i}.$$

Then f_n is Borel measurable and $f_n \to f$ pointwise; hence, f is Borel measurable on $\mathbb{R} \times \mathbb{R}^k$.

Proof. We first show that f is Borel measurable. Notice that, after fixing i, we get

$$f_{n,i}(x, y) = \frac{f(a_{i+1}, y)(x - a_i) - f(a_i, y)(x - a_{i+1})}{a_{i+1} - a_i}$$

is measurable. Hence, we can write

$$f_n(x, y) = \sum_{i \in \mathbb{Z}} f_{n,i}(x, y) \chi_{[a_i, a_{i+1}) \times \mathbb{R}^k}(x, y),$$

and so f_n is measurable.

Next, we need to show that $f_n \to f$ pointwise. Let $t = (x - a_i)/(a_{i+1} - a_i)$. Then we have that $f_n(x, y) = t f(a_{i+1}, y) + (1 - t) f(a_i, y)$. Using the continuity of f^y, we get that $n \to \infty$ implies $a_i, a_{i+1} \to x$, $t \to 0$, and so $f_n(x, y) \to f(x, y)$. □

Problem 100. If f is measurable, and $f = g$ μ almost everywhere, then g is measurable as long as μ is a complete measure.

Proof. Let $N = \{x : f(x) = g(x)\}$. Then $\mu(N^c) = 0$. If A is measurable, we need to show that $g^{-1}(A)$ is measurable. Notice that

$$g^{-1}(A) = (g^{-1}(A) \cap N) \sqcup (g^{-1}(A) \cap N^c) = (f^{-1}(A) \cap N) \sqcup (g^{-1}(a) \cap N^c).$$

We have $g^{-1}(A) \cap N^c \subset N^c$, $\mu(N^c) = 0$, so monotonicity tells us that the right part is measurable, and since f is measurable we have $f^{-1}(A) \cap N$ is measurable, so we get $g^{-1}(A)$ is measurable. □

Problem 101. Let $(X, \mathcal{M})$ be a measurable space.

(1) Prove that the Borel σ-algebra $\mathcal{B}_{\mathbb{C}}$ on $\mathbb{C}$ is generated by the open rectangles.
(2) Prove directly from the definitions that $f : X \to \mathbb{C}$ is $M - \mathcal{B}_{\mathbb{C}}$ measurable if and only if $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are measurable.
(3) Prove that the $\mathcal{M} - \mathcal{B}_{\mathbb{C}}$ measurable functions form a $\mathbb{C}$-vector space.
(4) Show that if $f : X \to \mathbb{C}$ is $\mathcal{M} - \mathcal{B}_{\mathbb{C}}$ measurable, then $|f| : X \to [0, \infty)$ is $\mathcal{M} - \mathcal{B}_{\mathbb{R}}$ measurable.
(5) Show that if (f_n) is a sequence of $\mathcal{M} - \mathcal{B}_{\mathbb{C}}$ measurable functions $X \to \mathbb{C}$ and $f_n \to f$ pointwise, then f is $\mathcal{M} - \mathcal{B}_{\mathbb{C}}$ measurable.

Proof. (1) Clear; use the topology given to $\mathbb{C}$ by $\mathbb{R}^2$.
(2) Since $\mathcal{B}_{\mathbb{C}}$ is generated by the open rectangles, we have f is measurable if and only if $f^{-1}(E)$, where E is an open rectangle, is measurable. Notice that we can view $f = \mathrm{Re}(f) \times \mathrm{Im}(f)$, so we have $f^{-1}(E) = \mathrm{Re}(f)^{-1}(E_1) \times \mathrm{Im}(f)^{-1}(E_2)$, where E_1 and E_2 are open balls in $\mathbb{R}^1$ and $E_1 \times E_2 = E$, an open rectangle. Hence, we have f is measurable if and only if $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are measurable.
(3) We need to show a few things. Denote the space of $\mathcal{M} - \mathcal{B}_{\mathbb{C}}$ measurable functions as V. Then if we have $f, g \in V$, we need to show $f + g \in V$. But we have $f + g = (\mathrm{Re}(f) + \mathrm{Re}(g)) + i(\mathrm{Im}(f) + \mathrm{Im}(g))$, and since addition of $\mathcal{M} - \mathcal{B}_{\mathbb{R}}$ measurable functions is again measurable, we get that $f + g$ are measurable using (2). Next, we need to show that for all $c \in \mathbb{C}$, $f \in V$, $cf \in V$. We can write $c = a + bi$. Then $cf = (a + bi)(\mathrm{Re}(f) + i\mathrm{Im}(f)) = (a\mathrm{Re}(f) - b\mathrm{Im}(f)) + i(a\mathrm{Im}(f) + b\mathrm{Re}(f))$, and since $\mathcal{M} - \mathcal{B}_{\mathbb{R}}$ is a vector space we get that $\mathrm{Im}(cf)$ and $\mathrm{Re}(cf)$ are both measurable, so using (2) gives that f is measurable. Hence, it is a vector space.
(4) Recall that we define $|f| = \sqrt{\mathrm{Re}(f)^2 + \mathrm{Im}(f)^2}$. Notice that f being $\mathcal{M} - \mathcal{B}_{\mathbb{R}}$ measurable implies f^2 is $\mathcal{M} - \mathcal{B}_{\mathbb{R}}$ measurable, and so we get $\mathrm{Re}(f)^2 + \mathrm{Im}(f)^2$ is $\mathcal{M} - \mathcal{B}_{\mathbb{R}}$ measurable. Finally, $\sqrt{\ }$ is continuous on the positive domain, which these are on, so we get that $|f|$ is $\mathcal{M} - \mathcal{B}_{\mathbb{R}}$ measurable.
(5) We have $\limsup_{n \to \infty} f_n = f$ is measurable if (f_n), f are real valued. We can write $f_n = \mathrm{Re}(f_n) + i\mathrm{Im}(f_n)$. Then $f_n \to f$ implies $\mathrm{Re}(f_n) \to \mathrm{Re}(f)$, $\mathrm{Im}(f_n) \to \mathrm{Im}(f)$. So using (2), we get that f is $\mathcal{M} - \mathcal{B}_{\mathbb{C}}$ measurable.

□

Problem 102. Show that almost uniform convergence implies convergence in measure.

Proof. Almost uniform convergence tells us that for all $\epsilon > 0$, there exists an $E \in \mathcal{M}$ such that $\mu(E) < \epsilon$ and $f_n \chi_{E^c} \to f \chi_{E^c}$ uniformly. Fix $\epsilon', \epsilon > 0$. Then we have $A \in \mathcal{M}$ as above, where $\mu(A) < \epsilon$. Hence, we have

$$\{|f_n - f| \geq \epsilon'\} = (\{|f_n - f| \geq \epsilon'\} \cap A) \sqcup (\{|f_n - f| \geq \epsilon'\} \cap A^c).$$

Notice as well that

$$\{|f_n - f| \geq \epsilon'\} \cap A \subset A,$$

and so it's measure is less than ϵ. Furthermore, since $f_n \to f$ uniformly on A^c, we can take N sufficiently large so that $\mu(\{|f_n - f| \geq \epsilon'\}) = 0$. Hence, we have

$$\mu(\{|f_n - f| \geq \epsilon'\}) < \epsilon.$$

Since the choice of $\epsilon > 0$ was arbitrary, we get

$$\mu(\{|f_n - f| \geq \epsilon'\}) = 0.$$

Since the choice of $\epsilon' > 0$ was arbitrary, we get that this holds for all $\epsilon' > 0$. ☐

Problem 103 (Folland 2.13)**.** Suppose $(f_N) \subset L^+$, $f_n \to f$ pointwise, and $\int f = \lim \int f_n < \infty$. Then $\int_E f = \lim \int_E f_n$ for all $E \in \mathcal{M}$. This need not be true if $\int f = \lim \int f_n = \infty$.

Proof. Notice we can write

$$\int_E f_n = \int f_n \chi_E.$$

Notice as well that Fatou's Lemma gives us

$$\int \liminf f_n \chi_E = \int f \chi_E = \int_E f \leq \liminf \int f_n \chi_E = \liminf \int_E f_n.$$

Similarly, we have

$$\int_{E^c} f \leq \liminf \int_{E^c} f_n.$$

Now, notice that

$$\int_E f = \int f - \int_{E^c} f \geq \int f - \liminf \int_{E^c} f_n = \limsup \left(\int f_n - \int_{E^c} f_n \right) = \limsup \int_E f_n.$$

So we have

$$\limsup \int_E f_n \leq \int_E f_n \leq \liminf \int_E f_n \implies \int_E f_n = \lim \int_E f_n.$$

For a counterexample, use $f = \chi_{(0,\infty)}$, $f_n = \chi_{(0,\infty} + n^2 \chi_{(-1/n,0]}$. ☐

Problem 104 (Folland 2.14)**.** If $f \in L^+$, let $\lambda(E) = \int_E f d\mu$ for $E \in \mathcal{M}$. Then λ is a measure on $\mathcal{M}$, and for any $g \in L^+$, $\int g d\lambda = \int f g d\mu$.

Proof. We show the first part. That is, λ is a measure. Clearly, we have $\lambda(\varnothing) = \int_\varnothing f d\mu = \int f \chi_\varnothing d\mu = 0$. Next, if E_n is a collection of disjoint sets, we have $\bigcup E_n = E$ is such that

$$\lambda(E) = \int_E f d\mu = \int f \chi_E d\mu = \int f \sum_{n=1}^{\infty} \chi_{E_n} d\mu = \sum_{n=1}^{\infty} \int f \chi_{E_n} d\mu = \sum_{n=1}^{\infty} \lambda(E_n).$$

So λ is indeed a measure. Next, we want to show that

$$\int g d\lambda = \int f g d\mu.$$

Let g be a simple function; i.e., $g = \sum_{n=1}^N a_n \chi_{E_n}$. Then we have

$$\int g d\lambda = \int \sum_{n=1}^{N} a_n \chi_{E_n} d\lambda = \sum_{n=1}^{N} a_n \int \chi_{E_n} d\lambda = \sum_{n=1}^{N} a_n \lambda(E_n) d\lambda = \sum_{n=1}^{N} a_n \int f \chi_{E_n} d\mu$$

$$= \int f g d\mu.$$

We can do this for all simple functions, then. Now, let g be a positive measurable function. Then we can construct a sequence $\psi_n \nearrow g$, ψ_n simple functions. Let $f_n = f \psi_n$. Then we have

$$\lim_{n \to \infty} \int \psi_n d\lambda = \int g d\lambda,$$

and

$$\lim_{n \to \infty} \int \psi_n d\lambda = \lim_{n \to \infty} \int f_n d\mu = \int f g d\mu.$$

Hence,

$$\int g d\lambda = \int f g d\mu.$$

□

Problem 105 (Folland 2.15)**.** If $(f_n) \subset L^+$, f_n decreasing pointwise to f, and $\int f_1 < \infty$, then $\int f = \lim \int f_n$.

Proof. We have $f_1 - f_n$ is increasing pointwise to $f_1 - f$, and $f_1 - f_n \in L^+$, and so we have the monotone convergence theorem gives

$$\lim \int (f_1 - f_n) = \int f_1 - \lim \int f_n = \int f_1 - \int f.$$

Since $\int f_1 < \infty$ and $f_1 \geq f_n \geq f$, we have that we can subtract everything from both sides to get

$$\lim \int f_n = \int f.$$

□

Problem 106 (Folland 2.16)**.** If $f \in L^+$ and $\int f < \infty$, for every $\epsilon > 0$ there exist $E \in \mathcal{M}$ such that $\mu(E) < \infty$ and $\int_E f > (\int f) - \epsilon$.

Proof. We can find a simple function $\psi \in \mathrm{SF}^+$ such that $0 \leq \psi \leq f$ and $\int \psi > \int f - \epsilon$. Notice that monotonicity of the integral says $\int \psi \leq \int f < \infty$, so $\int \psi < \infty$. Taking E to be the support of ψ, we get that

$$\int_E f \geq \int_E \psi = \int \psi > \int f - \epsilon.$$

□

Problem 107 (Folland 2.18)**.** Fatou's lemma remains valid if the hypothesis that $f_n \in L^+$ is replaced by the hypothesis that $f_n \in L^+$ is replaced by the hypothesis that f_n is measurable and $f_n \geq -g$ where $g \in L^+ \cap L^1$. What is the analogue of Fatou's lemma for nonpositive functions?

Proof. If we replace the hypothesis in Fatou's lemma as above, we get $f_n + g \geq 0$ for all n, and so writing $h_n = f_n + g$, we can use Fatou's Lemma to get

$$\int \liminf h_n \leq \liminf \int h_n.$$

Expanding h_n gives

$$\int \liminf f_n + \int g \leq \liminf \int f_n + \int g \leftrightarrow \int \liminf f_n \leq \liminf \int f_n,$$

since $g \in L^+ \cap L^1$. □

Problem 108 (Folland 2.19)**.** Suppose $(f_n) \subset L^1(\mu)$ and $f_n \to f$ uniformly.
(a) If $\mu(X) < \infty$, then $f \in L^1(\mu)$ and $\int f_n \to \int f$.
(b) If $\mu(X) = \infty$, the conclusion of (a) can fail.

Proof. (a) Notice that, since $f_n \to f$ uniformly, we can choose N sufficiently large so that $|f_n - f| \leq 1$ for all $x \in X$, $n \geq N$. Hence, we have

$$\int |f| \leq \int |f - f_N| + \int |f_N| \leq \mu(X) + \int |f_N| = \mu(X) + \int |f_N| < \infty.$$

So, we have $f \in L^1(\mu)$. We now want to leverage this to get that $\int f_n \to \int f$. To do so, we want to use the dominated convergence theorem. Let $g = 1 + |f|$. Then

$$\int g = \mu(X) + \int |f| < \infty,$$

and so we have $|f_n| \leq g$ for all $n \geq N$. Using the dominated convergence theorem, we get $\int f_n \to \int f$.

(b) Let $f_n(x) = \frac{1}{n}\chi_{(0,n)}$. Then

$$\int f_n(x) = 1$$

for all n, however $f_n(x) \to 0$ uniformly.

□

Problem 109 (Folland 2.20)**.** If $f_n, g_n, f, g \in L^1$, $f_n \to f$ and $g_n \to g$ a.e., $|f_n| \leq g_n$ and $\int g_n \to \int g$, then $\int f_n \to \int f$.

Proof. In this case, it suffices to check it for positive real functions. Hence, we have $g_n + f_n \geq 0$ and $g_n - f_n \geq 0$ for all n. Hence, we have

$$\int g + \int f \leq \liminf \int (g_n + f_n) = \liminf \int g_n + \liminf \int f_n.$$

Since $\int g_n \to g$, we have

$$\int g + \int f \leq \int g + \liminf \int f_n.$$

Notice as well that

$$\int g - \int f \leq \liminf \int (g_n - f_n) = \liminf \int g - \limsup \int f_n = \int g - \limsup \int f_n.$$

Hence, we have

$$\limsup \int f_n \leq \int f \leq \liminf \int f_n,$$

which says

$$\int f_n \to \int f.$$

□

Problem 110 (Folland 2.21)**.** Suppose $f_n, f \in L^1$ and $f_n \to f$ a.e. Then $\int |f_n - f| \to 0$ if and only if $\int |f_n| \to \int |f|$.

Proof. ($\implies$) Assume $\int |f_n - f| \to 0$. Then we have that the reverse triangle inequality gives

$$|f_n| - |f| \leq |f_n - f| \leftrightarrow |f_n| \leq |f_n - f| + |f|.$$

By the prior problem, this gives us that

$$\int |f_n| \to \int |f|.$$

($\impliedby$) We have

$$|f_n - f| \leq |f_n| + |f|,$$

so the prior problem tells us that

$$\int |f_n - f| \to 0$$

since

$$f_n \to f \text{ a.e.}$$

□

Problem 111 (Folland 2.22)**.** Use μ as the counting measure on $\mathbb{N}$ to interpret Fatou's lemma and monotone and dominated convergence theorem as statements about infinite series.

Proof. Fatou's Lemma: The original statement is

$$\int (\liminf f_n) d\mu \leq \liminf \int f_n d\mu.$$

We can interpret the integral as

$$\int f_n d\mu = \sum_{k=1}^{\infty} f_n(k),$$

and so we have

$$\int (\liminf f_n) d\mu = \sum_{k=1}^{\infty} \liminf_{n\to\infty} f_n(k) \leq \liminf_{n\to\infty} \sum_{k=1}^{\infty} f_n(k).$$

MCT:

$$\int f d\mu = \sum_{k=1}^{\infty} f(k) = \lim_{n\to\infty} \int f_n(x) d\mu = \lim_{n\to\infty} \sum_{k=1}^{\infty} f_n(k).$$

etc. □

Problem 112 (Folland 2.24)**.** Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$, and let $(X, \overline{\mathcal{M}}, \overline{\mu})$ be its completion. Suppose $f : X \to \mathbb{R}$ is bounded. Then f is $\overline{\mathcal{M}}$ measurable iff there exist sequence ϕ_n and ψ_n of $\mathcal{M}$-measurable simple functions such that $\phi_n \leq f \leq \psi_n$ and $\int (\psi_n - \phi_n) d\mu < n^{-1}$. In this case, $\lim \int \phi_n d\mu = \lim \int \psi_n d\mu = \int f d\overline{\mu}$.

Proof. ($\Longrightarrow$) Assume that f non-negative, bounded, and $\overline{\mathcal{M}}$ measurable. Let $g = f$ a.e., where g is $\mathcal{M}$ measurable. Let $N = \{x : |f(x) - g(x)| \neq 0\}$. Then using **Theorem 2.10b**, we can find simple functions $\phi_n \nearrow g$ and $\psi_n \searrow g$, where on N we let $\phi_n = 0$ and $\psi_n = M$, where $|f| \leq M$. Furthermore, using Chebychev, we have that

$$\int (\psi_n - \phi_n) d\mu < \mu(X)\epsilon$$

for n sufficiently large. Reorder the n to get the desired bound we want. Taking positive and negative parts respectively gives it for general f.
($\Longleftarrow$) Take $\Phi_n = \max\{\phi_1, \ldots, \phi_n\}$, $\Psi_n = \min\{\psi_1, \ldots, \psi_n\}$. □

Problem 113 (Folland 2.27)**.** Let $f_n(x) = ae^{-nax} - be^{-nbx}$ where $0 < a < b$. Show the following.

(a)

$$\sum_1^{\infty} \int_0^{\infty} |f_n(x)| dx = \infty.$$

(b)

$$\sum_1^{\infty} \int_0^{\infty} f_n(x) dx = 0.$$

(c)

$$\sum_1^{\infty} f_n \in L^1([0, \infty), m),$$

and

$$\int_0^{\infty} \sum_1^{\infty} f_n(x) dx = \log(b/a).$$

Proof. (a) Since we're taking the absolute value, we want to find the point c where

$$be^{-nbc} = ae^{-nac}.$$

This will give us the domain $(0, c)$ where $f_n < 0$ and (c, ∞) where $f_n > 0$. Solving gives

$$\frac{b}{a} = e^{nc(b-a)},$$
$$\log(b/a) = nc(b-a),$$
$$c = \frac{1}{n(b-a)}\log(b/a).$$

We can now write

$$\int_0^\infty |f_n| dx = -\int_0^c f_n dx + \int_c^\infty f_n dx.$$

This comes out to $\int_0^\infty |f_n|$ being proportional to $1/n$, and so it diverges.

□

Problem 114 (Folland 2.32)**.** Suppose $\mu(X) < \infty$. If f and g are complex-valued measurable functions on X, define

$$\rho(f, g) = \int \frac{|f-g|}{1+|f-g|} d\mu.$$

Then ρ is a metric on the space of measurable functions if we identify functions that are equal a.e., and $f_n \to f$ with respect to this measure if and only if $f_n \to f$ in measure.

Proof. We establish the first part; that is, ρ is a metric on this space of functions. To show it's a metric, we need to establish three things:

(1) $\rho(f, g) = 0$ if and only if $f = g$ a.e.:
($\implies$) If $\rho(f, g) = 0$, this wells us

$$\int \frac{|f-g|}{1+|f-g|} d\mu = 0.$$

Recall that $\int h d\mu = 0$ if and only if $h = 0$ a.e., so we have

$$\frac{|f-g|}{1+|f-g|} = 0$$

almost everywhere. Hence, we have $|f - g| = 0$ almost everywhere, which tells us that $f = g$ almost everywhere.
($\impliedby$) If $f = g$ almost everywhere, then clearly $|f - g| = 0$ almost everywhere and so we get $\rho(f, g) = 0$.

(2) $\rho(f, g) = \rho(g, f)$: This is clear by the symmetry of $|\cdot|$.

(3) $\rho(f, g) \le \rho(f, h) + \rho(h, g)$: By a prior homework problem, we know that

$$\frac{|f-g|}{1+|f-g|} \le \frac{|f-h|}{1+|g-h|} + \frac{|h-g|}{1+|h-g|}.$$

Integral respects monotonicity, so we get $\rho(f, g) \le \rho(f, h) + \rho(h, g)$, as desired.

For the next part, we need to establish that $f_n \to f$ with respect to this metric if and only if $f_n \to f$ in measure.

($\implies$) Assume $f_n \to f$ with respect to this metric. That is, for all $\epsilon > 0$, there exists an N such that for all $n \ge N$, we have $\rho(f_n, f) < \epsilon$. We want to establish that $f_n \to f$ in measure, which says that $\mu(\{|f_n - f| \ge \epsilon\}) \to 0$ for all $\epsilon > 0$. Notice that $\rho(f_n, f) < \epsilon$ implies that

$$\int \frac{|f_n - f|}{1+|f_n - f|} < \epsilon.$$

Notice as well that for all $\epsilon' > 0$, we have

$$\rho(f_n, f) = \int_{\{|f_n - f| \geq \epsilon'\}} \frac{|f_n - f|}{1 + |f_n - f|} + \int_{\{|f_n - f| < \epsilon'\}} \frac{|f_n - f|}{1 + |f_n - f|}.$$

So we have

$$\mu(\{|f_n - f| \geq \epsilon'\}) \left(\frac{\epsilon'}{1 + \epsilon'}\right) < \int_{\{|f_n - f| \geq \epsilon\}} \frac{|f_n - f|}{1 + |f_n - f|} < \epsilon.$$

Since we have that this inequality applies for all $\epsilon > 0$, we can take $\epsilon \to 0$ to get $\mu(\{|f_n - f| \geq \epsilon'\}) \to 0$, as desired.
($\Longleftarrow$) Assume $f_n \to f$ with respect to measure. Notice that we can write

$$\rho(f_n, f) = \int_{\{|f_n - f| \geq \epsilon'\}} \frac{|f_n - f|}{1 + |f_n - f|} + \int_{\{|f_n - f| < \epsilon'\}} \frac{|f_n - f|}{1 + |f_n - f|}.$$

We can bound this above by

$$\rho(f_n, f) \leq \mu(\{|f_n - f| \geq \epsilon'\}) + \epsilon' \mu(X).$$

Since $\mu(X) < \infty$ and the choice of ϵ' arbitrary, we can make the right hand side as small as possible using the convergence in measure. Hence,

$$\rho(f_n, f) \to 0,$$

as desired. □

Problem 115 (Folland 2.33)**.** If $f_n \geq 0$ and $f_n \to f$ in measure, then

$$\int f \leq \liminf \int f_n.$$

Proof. We can use **Theorem 2.30**. Notice that we have a subsequence $f_i \to \liminf f_n$. Furthermore, we get $f_i \to f$ in measure as well. So we can take a subsequence of this by the theorem, denoted by f_{i_j}, which converges to f a.e. So Fatou's lemma gives

$$\int f = \int \lim_{j \to \infty} f_{i_j} \leq \liminf_{j \to \infty} \int f_{i_j} = \liminf_{n \to \infty} \int f_n.$$

□

Problem 116 (Folland 2.34)**.** Suppose $|f_n| \leq g \in L^1$ and $f_n \to f$ in measure.
(a)

$$\int f = \lim \int f_n.$$

(b) $f_n \to f$ in L^1.

Proof. (a) Using the prior problem, we have

$$\int f \leq \liminf \int f_n.$$

Notice that $g - f_n \geq 0$, $g + f_n \geq 0$, and so we can use this to get

$$\int (g - f) \leq \liminf \int (g - f_n) = \int g - \limsup \int f_n,$$

$$\int (g + f) \leq \int g + \liminf \int f_n.$$

Hence, we have

$$\limsup \int f_n \leq \int f \leq \liminf \int f_n,$$

as desired.

(b) Since $f_n \to f$ in measure, we have for all $\epsilon > 0$
$$\mu(\{|f_n - f| \geq \epsilon\}) \to 0.$$
Notice that $f_n \to f$ in measure implies $|f_n - f| \to 0$ in measure. Notice as well that $|f_n - f| \leq |f_n| + |f| \leq g + |f|$. Since $g + |f| \in L^1$, we can use (a) to get that
$$\int 0 = 0 = \lim \int |f_n - f|,$$
which gives convergence in L^1.

□

Problem 117 (Folland 2.35)**.** We have $f_n \to f$ in measure if and only if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that
$$\mu(\{|f_n - f| \geq \epsilon\}) < \epsilon$$
for $n \geq N$.

Proof. ($\implies$) Follows from the definition.
($\impliedby$) Assuming the condition, we want to show that $\mu(\{|f_n - f| \geq \epsilon\}) \to 0$ for all $\epsilon > 0$. Notice that for $\epsilon' < \epsilon$, we have
$$\{|f_n - f| \geq \epsilon\} \subset \{|f_n - f| \geq \epsilon'\},$$
so
$$\mu(\{|f_n - f| \geq \epsilon\}) \leq \mu(\{|f_n - f| \geq \epsilon'\}) < \epsilon'.$$
Since this applies for all $\epsilon' < \epsilon$, we get that it goes to 0. Hence, we have convergence in measure. □

Problem 118 (Folland 2.36)**.** If $\mu(E_n) < \infty$ and $\chi_{E_n} \to f$ in L^1, then f is a.e. equal to the characteristic function of a measurable set.

Proof. Since it converges in L^1, we have
$$\int |\chi_{E_n} - f| \to 0.$$
Notice this implies convergence in measure, and so we have
$$\mu(\{|\chi_{E_n} - f| \geq \epsilon\}) \to 0.$$
Using the theorem from the book, we have there is a subsequence $\chi_{E_{n_j}}$ which converges to f a.e., which means that f is equal to a characteristic function of a measurable set a.e. □

Problem 119. Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a right continuous, increasing, bounded function. Show that
$$\int_{\mathbb{R}} (\alpha(x+c) - \alpha(x))dx = c \int_{\mathbb{R}} d\alpha \quad \text{for each } c > 0,$$
where $\int f dx$ is the Lebesgue integral and $\int f d\alpha$ is the Lebesgue-Stieltjes integral.

Proof. Notice that
$$\alpha(x+c) - \alpha(x) = \mu_\alpha((x, x+c]),$$
so we have
$$\int_{\mathbb{R}} (\alpha(x+c) - \alpha(x))dx = \int_{\mathbb{R}} \mu_\alpha((x, x+c])dx = \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_{(x,x+c])} d\alpha dx.$$
Invoke Tonelli to get this is equal to
$$\int_{\mathbb{R}} \int_{\mathbb{R}} \chi_{(x,x+c]} dx d\alpha = \int_{\mathbb{R}} c d\alpha$$
as desired. □

Problem 120. Suppose μ is a positive measure on $(X, \mathcal{M})$ and $f \in L^1(X)$.

(1) Prove that if $E \subset X$ with $\mu(E) = 0$, then $\int_E f d\mu = 0$.
(2) Prove that if $\int_E f d\mu = 0$ for all $E \in \mathcal{M}$, then $f = 0$ μ-a.e.

Proof. (1) Assume that $f \geq 0$. We can construct a sequence of simple measurable functions ϕ_n such that $\phi_n \nearrow f$. Assume first that $f = \chi_F$, where F is some measurable set. Then

$$\int_E \chi_F d\mu = \mu(E \cap F) \leq \mu(E) = 0.$$

So it holds for characteristic functions. By linearity, this extends to simple measurable functions, and applying the monotone convergence theorem, we have

$$0 = \lim_n \int_E \phi_n d\mu = \int_E f d\mu.$$

In the case where f is not strictly positive, write $f = f_+ - f_-$ and use linearity again to get that it holds for these functions.

(2) We proceed by the contrapositive; assume $f \neq 0$ μ-a.e. Then $F = \{f \neq 0\}$ is such that $\mu(F) > 0$. Furthermore, let $F = F_1 \sqcup F_2$, $F_1 = \{f > 0\}$, $F_2 = \{f < 0\}$. Then we have that

$$\int_F f d\mu = \int_{F_1} f d\mu + \int_{F_2} f d\mu.$$

Notice that since $\mu(F) >)$, we must have at least one of $\mu(F_1), \mu(F_2) > 0$. Assume without loss of generality it's F_1, then we get that

$$\int_{F_1} f d\mu > 0.$$

Thus, we have that there exists a set $E \in \mathcal{M}$ so that $\int_E f d\mu \neq 0$.

□

Problem 121. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Let f and g be real-valued integrable functions such that $\int_X f d\mu = \int_X g d\mu$. Prove that either $f = g$ a.e., or there exists an $E \in \mathcal{M}$ such that $\int_E f d\mu > \int_E g d\mu$.

Proof. Assume $f \neq g$ a.e., $\int_X f = \int_X g$. Then we have that there is a measurable F so that $F = \{f \neq g\}$ and $\mu(F) > 0$. Write $F_1 = \{f > g\}$, $F_2 = \{f < g\}$. Then we must have that $\mu(F_1) > 0$ or $\mu(F_2) > 0$, since $\mu(F) = \mu(F_1) + \mu(F_2)$. Assume for contradiction that $\mu(F_1) = 0$. Then we have that

$$\int_X f = \int_X g = \int_{F_2} g + \int_{F^c} g = \int_{F_2} f + \int_{F^c} f.$$

Since $\mu(X) < \infty$, f, g integrable, it's fine to move things around to get

$$\int_{F_2} g = \int_{F_2} f.$$

In other words,

$$\int_{F_2} (f - g) = 0.$$

But this can happen only if $f = g$ a.e. on F_2, which force $\mu(F_2) = 0$, a contradiction. Hence, we must have $\mu(F_1) > 0$, and so there is a measurable set so that

$$\int_{F_1} f > \int_{F_1} g.$$

□

Problem 122. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(1) Suppose $\mu(X) < \infty$. Let $f : X \to \mathbb{R}$ be measurable, (f_n) a sequence of measurable functions, and suppose $f_n \to f$ in μ-measure. Let $g : \mathbb{R} \to \mathbb{R}$ be continuous. Let $h = g \circ f$, and for each n let $h_n = g \circ f_n$. Prove that $h_n \to h$ in measure.
(2) Show by an example that finiteness cannot be dropped.

Proof. (1) Recall that convergence in measure means that, for all $\epsilon > 0$, we have that

$$\mu(|h_n - h| \geq \epsilon) \to 0 \text{ as } n \to \infty.$$

Since $f_n \to f$ in measure, we can find a subsequence $f_{n_k} \to f$ almost everywhere. Hence, $h_{n_k} \to h$ almost everywhere, since g is continuous.

(2)

□

Problem 123 (Folland 2.37)**.** Suppose that f_n and f are measurable complex-valued funtions and $\phi : \mathbb{C} \to \mathbb{C}$.

(a) If ϕ is continuous and $f_n \to f$ a.e., then $\phi \circ f_n \to \phi \circ f$ a.e.
(b) If ϕ is uniformly continuous and $f_n \to f$ uniformly, almost uniformly, or in measure, then $\phi \circ f_n \to \phi \circ f$ uniformly, almost uniformly, or in measure, respectively.
(c) There are counterexamples when the continuity assumptions on ϕ are not met.

Proof. (a) We have $f_n \to f$ a.e. implies that $E = \{f_n \to f\}$ is such that $\mu(E^c) = 0$. We want to show that $F = \{x \in X \ : \ \phi \circ f_n(x) \to \phi \circ f(x)\}$ is such that $\mu(F^c) = 0$. Notice that, since ϕ is continuous, we have

$$\lim_{n\to\infty} \phi(f_n(x)) = \phi(f(x)).$$

Hence, we get $E \subseteq F$, which implies $F^C \subseteq E^C$, and so $\mu(F^C) = 0$.

(b) For uniformly, we see that $f_n \to f$ uniformly implies that for all $\epsilon > 0$, there is an N such that for all $n \geq N$,

$$|f_n - f| < \epsilon.$$

Since ϕ is uniformly continuous, we have that for all $\epsilon > 0$, there exists a δ such that

$$|x - y| < \delta \implies |\phi(x) - \phi(y)| < \epsilon.$$

We then see that we can choose N sufficiently large so that $|f_n - f| < \delta$, which implies that $|\phi \circ f_n - \phi \circ f| < \epsilon$. Hence, we have that $\phi \circ f_n \to \phi \circ f$ uniformly. Almost uniformly is the same argument. For measure, we see that we have

$$\mu(\{|f_n - f| \geq \epsilon\}) \to 0.$$

Notice that uniform continuity tells us that

$$|\phi(x) - \phi(y)| \geq \epsilon \implies |x - y| \geq \delta.$$

Hence,

$$\{|\phi \circ f_n - \phi \circ f| \geq \epsilon\} \subset \{|f_n - f| \geq \delta\},$$

and so

$$\mu(\{|\phi \circ f_n - \phi \circ f| \geq \epsilon\}) \to 0.$$

Hence, it converges in measure.

(c) Omitted

□

Problem 124 (Lecture notes)**.** Consider the canonical projections π_X, π_Y on $X \times Y$.

(1) If X, Y topological spaces, prove π_X, π_Y are open maps.
(2) If these are measurable spaces, do they map measurable sets to measurable sets?

Proof. (1) We need to show that $\pi_X(G) \in \tau$ for all $G \in \tau \times \theta$. Since $G \in \tau \times \theta$, we have $G = \bigcup_i \bigcap_j F_{i,j} \times H_{i,j}$, where $F_{i,j} \in \tau$, $H_{i,j} \in \theta$. Hence,

$$\pi_X \left(\bigcup_i \bigcap_j F_{i,j} \times H_{i,j} \right) = \bigcup_i \pi_X \left(\bigcap_j F_{i,j} \times \bigcap_j H_{i,j} \right) = \bigcup_i \bigcap_j F_{i,j} \in \tau,$$

where the intersections are finite.

(2) The answer is no. See below. □

Problem 125 (Lecture notes)**.** Use the proposition to prove that $\mathcal{L} \times \mathcal{L}$ is not equal to $\mathcal{L}^2 := (\lambda \times \lambda)^*$ measurable sets.

Proof. Let V be the Vitali set in $[0,1]$, and let $\{x\}$ be some point. Take some E mble such that $V \subset E$. Then, since λ is a complete measure, we have

$$(\lambda \times \lambda)^*(E \times \{x\}) = 0 \implies (\lambda \times \lambda)^*(V \times \{x\}) = 0,$$

and so $V \times \{x\}$ is measurable. However, $\pi^x(V \times \{x\}) = V$ is not a measurable set, and so we cannot have $\mathcal{L} \times \mathcal{L} = \mathcal{L}^2$. □

Problem 126 (Folland 2.40)**.** In Egoroff's theorem, the hypothesis "$\mu(X) < \infty$" can be replaced by $|f_n| \leq g$ for all n, where $g \in L^1(\mu)$."

Proof. We follow the proof. We have $f_n \to f$ almost everywhere on X. For $k, n \in \mathbb{N}$ let

$$E_n(k) = \bigcup_{m=n}^{\infty} \{x \ : \ |f_m(x) - f(x)| \geq k^{-1}\}.$$

Then for fixed k, $E_n(k)$ decreases as n increases, and $\bigcap_{n=1}^{\infty} E_n(k) = \varnothing$. Now, we have that $|f_n| \leq g$. Notice that this gives us $|f_n - f| \leq |f_n| + |f| \leq g + |f|$. Since $|f_n| \leq g$ for all n, we have $|f| \leq g$, so we can write this as $|f_n - f| \leq 2g$. Now, let

$$A(k) := \{x \ : \ 2|g| \geq k^{-1}\}.$$

We have that

$$E_1(k) \subset A(k).$$

Furthermore,

$$\infty > 2 \int |g| \geq 2 \int_{A(k)} |g| \geq 2 \int_{A(k)} 1/k = 2\mu(A(k))1/k^{-1}.$$

So for fixed k, we have $\mu(A(k)) < \infty$. Hence, $\mu(E_1(k)) < \infty$. Thus, as in the original proof, we can conclude that $\mu(E_n(k)) \to 0$ as $n \to \infty$. The rest of the proof is the same. □

Problem 127 (Folland 2.42)**.** Let μ be counting measure on $\mathbb{N}$. Then $f_n \to f$ in measure if and only if $f_n \to f$ uniformly.

Proof. ($\Longrightarrow$) We have $f_n \to f$ in measure implies $\mu(\{x \in \mathbb{N} \ : \ |f_n(x) - f(x)| \geq \epsilon\}) \to 0$ as $n \to \infty$. Since μ is counting measure, this implies that for N sufficiently large, we have that for all $n \geq N$,

$$\mu(\{x \in \mathbb{N} \ : \ |f_n(x) - f(x)| \geq \epsilon\}) = 0.$$

In $\mathbb{N}$, the only μ-null set is $\varnothing$, so this implies that $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$. Hence, we have uniform convergence; for all $\epsilon > 0$, there exists an N such that for all $n \geq N$, we have $|f_n(x) - f(x)| < \epsilon$.

($\Longleftarrow$) Fix $\epsilon > 0$. Then there is an N such that for all $n \geq N$, $|f_n - f| < \epsilon$. Hence, we have

$$\mu(\{|f_n - f| \geq \epsilon\}) = 0$$

for N sufficiently large. Thus,

$$\lim_{n\to\infty} \mu(\{|f_n - f| \geq \epsilon\}) = 0.$$

The choice of $\epsilon > 0$ was arbitrary, and so we have convergence in measure. $\square$

Problem 128 (Folland 2.47)**.** Let $X = Y$ be an uncountable linearly ordered set such that for each $x \in X$, $\{y \in X \ : \ y < x\}$ is countable. Let $\mathcal{M} = \mathcal{N}$ be the σ algebra of the countable or cocountable sets, and let $\mu = \nu$ be defined on $\mathcal{M}$ by $\mu(A) = 0$ if A is countable and $\mu(A) = 1$ if A is cocountable. Let

$$E = \{(x, y) \in X \times X \ : \ y < x\}.$$

Then E_x and E^y are measurable for all x, y, and

$$\int\int \chi_E d\mu d\nu$$

and

$$\int\int \chi_E d\nu d\mu$$

exist but are not equal.

Proof. Fix x. Then we have that $E_x = \{y \ : \ (x, y) \in E\} = \{y \ : \ y < x\}$ is countable by assumption, so it is measurable. Fix y. Then we have that $E^y = \{x \ : \ (x, y) \in E\} = \{x \ : \ y < x\}$. Notice that the complement $(E^y)^C = \{x \ : \ y > x\} \cup \{y\}$ is countable, and so E^y is cocountable. Hence, it is measurable.

Now, we wish to compute

$$\int\int \chi_E d\mu d\nu.$$

Notice that

$$\int \chi_{E^y} d\mu(x) = \mu(E^y) = 1,$$

and so

$$\int\int \chi_E d\mu d\nu = \int d\nu(y) = \nu(Y) = 1.$$

Similarly, we get

$$\int \chi_{E_x} d\nu(x) = \nu(E_x) = 0,$$

and so

$$\int\int \chi_E d\nu d\mu = \int 0 d\mu = 0.$$

Thus, they are unequal. $\square$

Problem 129 (Folland 2.49)**.** Prove **Theorem 2.39** in Folland in the following way:

(1) If $E \in M \times N$ and $(\mu \times \nu)(E) = 0$, then $\nu(E_x) = \mu(E^y) = 0$ for a.e. x and y.
(2) If f is $\mathcal{L}$ measurable and $f = 0$ λ a.e., then f_x and f^y are integrable for a.e. x and y. Furthermore, $\int f_x d\nu = \int f^y d\mu = 0$ for almost every x and y.
(3) Use Proposition 2.12 and finish the proof.

Proof. (1) We have

$$\int \chi_E d(\mu \times \nu) = (\mu \times \nu)(E).$$

Notice that Tonelli's theorem gives

$$\int \chi_E d(\mu \times \nu) = (\mu \times \nu)(E) \int \left(\int \chi_{E^y}(x) d\mu(x)\right) d\nu(y) = 0.$$

This then tells us that
$$\int \chi_{E^y}(x)d\mu(x) = \mu(E^y) = 0$$
for almost every y. We have a similar argument for $\nu(E_x)$.

(2) Let $F = \{(x, y) \in X \times Y \ : \ f(x, y) \neq 0\}$. We have $(\mu \times \nu)(F) = 0$, and so we can find $F \subset E$ so that $E \in \mathcal{L}$ and $(\mu \times \nu)(E) = 0$. From (1), we have $\mu(E^y) = 0$, $\nu(E_x) = 0$, and so $\mu(F^y) = 0$ and $\nu(F_x) = 0$. Finally, we see that
$$\int |f_x| d\nu(y) = \int \chi_{F_x}(y)|f_x(y)|d\nu(y) = 0,$$
so f_x is integrable for a.e. x, and likewise for f^y. Furthermore, this tells us that their integrals are 0.

(3) **Proposition 2.12** tells us that we have a $\widetilde{f} = f$ a.e., where $\widetilde{f}$ is $\mathcal{M} \times \mathcal{N}$ measurable. We now use (1) and (2) to deduce that Tonelli/Fubini's theorem still applies.

□

Problem 130 (Folland 2.50)**.** Suppose $(X, \mathcal{M}, \mu)$ is a σ-finite measure space and $f \in L^+(X)$. Let
$$G_f = \{(x, y) \in X \times [0, \infty] \ : \ y \leq f(x)\}.$$
Show that G_f is $\mathcal{M} \times \mathcal{B}_{\mathbb{R}}$ measurable and $\mu \times m(G_f) = \int f d\mu$.

Proof. As in the hint, we have that $(x, y) \mapsto (f(x), y)$ is measurable, since the pullback of an open set $E \times F$ will be $f^{-1}(E) \times F$, which is measurable using the measruablitiy of f.

Next, we see that the map $(x, y) \mapsto x - y$ is measurable, since subtraction is continuous and so measurable. Then $g(x, y) = f(x) - y$, which is their composition, is measurable, and so notice that $G_f = \{(x, y) \ : 0 \leq f(x) - y\} = \{\infty > g \geq 0\}$ which is measurable.

Let $h = \chi_{G_f}$. Then we have that h is measurable, since G_f is measurable, and furthermore we have that Tonelli gives
$$\int \chi_{G_f} d(\mu \times m) = \int \left(\int \chi_{G_f} dm \right) d\mu.$$
Notice that
$$\int \chi_{G_f}(x, y) dm(y) = m((G_f)_x),$$
where
$$(G_f)_x = \{y \in Y \ : \ (x, y) \in G_f\} = \{y \in [0, \infty] \ : \ y \leq f(x)\}.$$
Hence, we have
$$m((G_f)_x) = m([0, f(x)]) = f(x),$$
and so
$$\int \left(\int \chi_{G_f} dm \right) d\mu = \int f d\mu.$$

□

Problem 131 (Folland 2.51)**.** Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be arbitrary measure spaces.

(1) If $f : X \to \mathbb{C}$ is $\mathcal{M}$-measurable, $g : Y \to \mathbb{C}$ is $\mathcal{N}$ measurable, and h is defined on $X \times Y$ by $h(x, y) = f(x)g(y)$, then h is $\mathcal{M} \times \mathcal{N}$ measurable.

(2) If $f \in L^1(\mu)$, $g \in L^1(\nu)$, then $h \in L^1(\mu \times \nu)$ and
$$\int h d(\mu \times \nu) = \int f d\mu \int g d\nu.$$

Proof. (1) Let $F(x,y) := f(x)$, $G(x,y) := g(y)$. We have that F is $\mathcal{M}\times\mathcal{N}$ measurable; taking $E \subset \mathbb{C}$ open, we have $F^{-1}(E) = f^{-1}(E)\times Y$, and since f is $\mathcal{M}$ measurable we have that this is measurable. The same argument applies to G. Notice that $h(x,y) = F(x,y)G(x,y)$. Since the product of measurable functions is measurable, we have that h is measurable.

(2) Notice that

$$\int |h| d(\mu\times\nu) = \int |F(x,y)||G(x,y)| d(\mu\times\nu).$$

Since we're in L^+, we can use Tonelli to get

$$\int |h| d(\mu\times\nu) = \left(\int |g(y)| d\nu(y)\right)\left(\int |f(x)| d\mu(x)\right) < \infty,$$

since $f, g \in L^1$. Now, since h is integrable, we apply Fubini to get

$$\int f(x)g(y) d(\mu\times\nu) = \int f(x) d\mu(x) \int g(y) d\nu(y).$$

□

Problem 132 (Folland 2.56)**.** If f is Lebesgue integrable on $(0,a)$ and $g(x) = \int_x^a t^{-1} f(t)dt$, then g is integrable on $(0,a)$ and $\int_0^a g(x)dx = \int_0^a f(x)dx$.

Proof. Notice that

$$\int_x^a t^{-1} f(t)dt = \int_0^a \chi_{(x,a)}(t) t^{-1} f(t)dt,$$

and so

$$\int_0^a |g(x)|dx \le \int_0^a \int_0^a \chi_{(x,a)}(t) t^{-1} |f(t)| dt dx.$$

We use Fubini/Tonelli to get that this is equal to

$$\int_0^a \int_0^a \chi_{(x,a)}(t) t^{-1} |f(t)| dt dx = \int_0^a \left(\int_0^a \chi_{(x,a)}(t) dx\right) t^{-1} |f(t)| dt.$$

Notice that

$$\int_0^a \chi_{(x,a)}(t) dx$$

is 1 if $t \in (x,a)$ and 0 if $t \notin (x,a)$. So we get that it's equal to

$$\int_0^a \chi_{(x,a)}(t) dx = \int_0^a \chi_{(0,t)}(x) dx = t.$$

So we get

$$\int_0^a |g(x)|dx \le \int_0^a (t) t^{-1} |f(t)| dt = \int_0^a |f(t)| dt < \infty.$$

So it's integrable on $(0,a)$, and we go back and apply Fubini to get desired result.

□

Problem 133.

(1) Let E be a measurable subset of $\mathbb{R}^2$ such that for almost every x, $E_x = \{y : (x,y) \in E\}$ has $\mathbb{R}$ measure zero. Show that E has measure zero, and that for almost every $y \in \mathbb{R}$, $E^y = \{x : (x,y) \in E\}$ has measure zero.

(2) Let $f(x,y)$ be non-negative and measurable in $\mathbb{R}^2$. Suppose that for almost every $x \in R$, $f(x,y) = f_x$ is finite for almost every y. Show that for almost every $y \in \mathbb{R}$, $f(x,y)$ is finite for almost every x.

Proof.

(1) Examine χ_E. We have that

$$\mu(E) = \int \chi_E d(\mu \times \mu).$$

Tonelli's theorem gives us that

$$\int \chi_E d(\mu \times \mu) = \int \left(\int \chi_{E_x}(y) d\mu(y) \right) d\mu(x) = \int \mu(E_x) d\mu(x) = 0.$$

Hence, E has measure zero. Notice as well this tells us that $\mu(E^y) = 0$.

(2) Let $E = \{(x, y) \ : \ f(x, y) = \infty\}$. Then $E_x = \{y \ : \ f_x(y) = \infty\}$ has measure zero, and so by the prior problem we get E has measure zero and E^y has measure zero.

□

Problem 134. Show that $f * g = g * f$, assuming the integral in question exists.

Proof. We have

$$(f * g)(x) = \int f(x - y) g(y) dy.$$

Let $z = x - y$, then we have

$$\int f(x - y) g(y) dy = \int f(z) g(x - z) dz = \int g(x - z) f(z) dz = (g * f)(x).$$

□

Problem 135. Show that $(f * (g * h))(x) = ((f * g) * h)(x)$, assuming the integral exists.

Proof. Notice that

$$(f * (g * h))(x) = \int f(x - y)(g * h)(y) dy = \int f(x - y) \left(\int g(y - z) h(z) dz \right) dy.$$

By Fubini, we have

$$\int \int f(x - y) g(y - z) h(z) dz dy.$$

Let $t = x - y$, then we can rewrite this as

$$\int \int f(t) g(x - t - z) h(z) dz dt.$$

Since $f * (g * h) = (g * h) * f$, we have

$$\int \int f(t) g(x-t-z) h(z) dt dz = \int h(z) \int f(t) g(x-t-z) dt dz = \int h(z)(g*f)(x-z) dz = h*(g*f)(x),$$

and using commutativity again we have

$$(h * (g * f))(x) = ((f * g) * h)(x).$$

Hence,

$$f * (g * h) = (f * g) * h.$$

□

Problem 136 (Royden 17.1.2)**.** Let $\mathcal{M}$ be a σ-algebra of sets of X, and the set function $\mu : \mathcal{M} \to [0, \infty)$ be finitely additive. Prove that μ is a measure if and only if it satisfies continuity from below.

Proof. ($\implies$) Assume μ is a measure. Then we wish to show that if $E_1 \subset E_2 \subset \cdots$ is a sequence of increasing sets, we have

$$\mu\left(\bigcup E_i\right) = \lim \mu(E_n).$$

We can disjointify E_i to get

$$F_n = E_n - \left(\bigcup_{i=1}^{n-1} E_i\right),$$

$$\bigcup F_i = \bigcup E_i,$$

$$\mu\left(\bigcup F_i\right) = \mu\left(\bigcup E_i\right) = \sum \mu(F_i).$$

Furthermore, we see that

$$\sum_{i=1}^{n} \mu(F_i) = \mu(E_n);$$

going by induction, assuming it holds for $n-1$, we have

$$\sum_{i=1}^{n} \mu(F_i) = \sum_{i=1}^{n-1} \mu(F_i) + \mu(F_n) = \mu(E_{n-1}) + \mu(F_n).$$

Since

$$E_n = E_{n-1} \sqcup E_n - E_{n-1},$$

we get that

$$\mu(E_n) = \mu(E_{n-1}) + \mu(F_n),$$

and so we have

$$\mu\left(\bigcup E_i\right) = \lim \mu(E_n).$$

($\impliedby$) Assume that it satisfies continuity from below. Since it's finitely additive, notice that

$$\mu(\varnothing) = 2\mu(\varnothing),$$

and since its a finite additive function we have that this means it's 0. Next, we need to show that if E_i is a disjoint collection of sets, we have

$$\mu\left(\bigcup E_i\right) = \sum \mu(E_i).$$

Let $F_n = \bigcup_{i=1}^{n} E_i$. Then we have that F_n is an increasing sequence of sets, and furthermore

$$\bigcup F_n = \bigcup E_n.$$

Hence, we have

$$\mu\left(\bigcup E_i\right) = \mu\left(\bigcup F_i\right) = \lim \mu(F_n).$$

Using the fact that E_i are disjoint and μ is finitely additive, we have

$$\mu(F_n) = \sum_{i=1}^{n} \mu(E_i),$$

and so we get

$$\mu\left(\bigcup E_i\right) = \sum \mu(E_i),$$

as desired. ☐

Problem 137 (Royden 17.1.11)**.** Let μ nad ν be measure on a measurable space $(X, \mathcal{M})$. For $E \in \mathcal{M}$, define $\zeta(E) = \max\{\mu(E), \nu(E)\}$. Is ν a measure?

Proof. To be a measure, we need to satisfy two properties.

(1) Notice that $\zeta(\varnothing) = \max(0, 0) = 0$.

(2) Let $A = \{0, 1\}$. Define $\mu(\{0\}) = \nu(\{1\}) = 1$, and $\mu(\{1\}) = \nu(\{0\}) = 0$. Then
$$\zeta(\{1\}) = 1, \zeta(\{0\}) = 1,$$
but
$$\zeta(A) = 1 \neq 2 = \zeta(\{0\}) + \zeta(\{1\}).$$

□

Problem 138 (Royden 17.4.19)**.** Show that any measure that is induced by an outer measure is complete.

Proof. To be complete, we need that $\mu(N) = 0 \implies N \in \mathcal{M}$. If μ^* is an outer measure, we have that it is a function $\mathcal{P}(X) \to [0, \infty]$ which satisfies properties of being an outer measure. Then μ is a measure if we restrict it to the μ^* measurable sets, which are the sets A that satisfy for all $E \subset X$,
$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$
Let F be a set where $\mu^*(F) = 0$. Then
$$\mu^*(E \cap F) + \mu^*(E \cap F^c) \leq \mu^*(E \cap F^c) \leq \mu^*(E),$$
$$\mu^*(E) \leq \mu^*(E \cap F) + \mu^*(E \cap F^c),$$
and so
$$\mu^*(E) = \mu^*(E \cap F) + \mu^*(E \cap F^c).$$
Hence, $F \in \mathcal{M}$ is measurable. So the measure is complete. □

Problem 139 (Royden 17.5.29)**.** Show that a set function on a σ-algebra is a measure if and only if it is a premeasure.

Proof. Recall that to be a measure, we require two things:

(1) $\mu(\varnothing) = 0$,

(2) If E_i is a disjoint collection of mble sets, then
$$\mu\left(\bigcup E_i\right) = \sum \mu(E_i).$$

To be a premeasure on an algebra, we require that:

(1) $\mu(\varnothing) = 0$,

(2) If E_i is a disjoint collection of sets in the algebra, and $\bigcup E_i$ in the algebra, then
$$\mu\left(\bigcup E_i\right) = \sum \mu(E_i).$$

($\implies$) If μ is a measure on $\mathcal{M}$, the σ-algebra, then we have that it is clearly a premeasure.
($\impliedby$) If μ is a premeasure on the σ-algebra $\mathcal{M}$, notice that for any disjoint collection of sets $E_i \in \mathcal{M}$, we have $\bigcup E_i \in \mathcal{M}$, and so we have that it is a measure. □

Problem 140 (Royden 18.1.2)**.** Suppose $(X, \mathcal{M}, \mu)$ is not complete. Let E be a subset of a set of measure zero that does not belong to $\mathcal{M}$. Let $f = 0$ on X and $g = \chi_E$. Show that $f = g$ a.e. on X, while f is measurable and g is not.

Proof. We first show that $f = g$ a.e. Notice that $\{x : f(x) = g(x)\} = E^c$, and since E has null measure we get that $f = g$ a.e. Next, we see that f is measurable, $\{f > a\}$ is either nothing or the whole set depending on a, and so is measurable either way. We see that g is not, since $\{g > 1/2\} = E$, which is not measurable by assumption. □

Problem 141 (Royden 18.1.4)**.** Let E be a measurable subset of X and f an extended real-valued function on X. Show that f is measurable if and only if its restriction to E and $X - E$ are measurable.

Proof. ($\implies$) We see that, for all open U, we have $f^{-1}(U) \cap E$ is measurable and $f^{-1}(U) \cap (X - E)$ is measurable, since f and E are measurable.
($\impliedby$) Notice that, for U open, $f^{-1}(U) = (f^{-1}(U) \cap E) \sqcup (f^{-1}(U) \cap E^c)$. Since each of these are measurable, we have $f^{-1}(U)$ is measurable. □

Problem 142 (Royden 18.1.6)**.** Consider two extended real-valued measurable functions f, g on X that are finite a.e. on X. Define X_0 to be the set of points in X at which both f and g are finite. Show that X_0 is measurable and $\mu(X - X_0) = 0$.

Proof. Let $A = \{|f| < \infty\}$ and $B = \{|g| < \infty\}$, then $X_0 = A \cap B$. Furthermore, since f, g are measurable, we have that A and B are measurable, since $A = \{f > -\infty\} \cap \{f < \infty\}$ and likewise for B. Hence, X_0 is an intersection of measurable sets, and so measurable.

Notice as well that $(X_0)^c = (A \cap B)^c = A^c \cup B^c$. By subadditivity, we have

$$\mu((X_0)^c) \leq \mu(A^c) + \mu(B^c) = 0.$$

□

Problem 143 (Borel-Cantelli Lemma)**.** If

$$\sum \mu(E_i) < \infty,$$

then

$$\mu\left(\limsup E_i\right) = 0.$$

Proof. Recall that

$$\limsup E_i = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} E_m.$$

Notice as well that

$$\sum \mu(E_i) < \infty \implies \lim_{n\to\infty} \sum_{m=n}^{\infty} \mu(E_m) = 0;$$

that is, the tails converge to 0. Using continuity from above, and the fact that $\mu(E_1) < \infty$, we have

$$\mu(\limsup E_i) = \lim_{n\to\infty} \mu\left(\bigcup_{m=n}^{\infty} E_m\right) \leq \lim_{n\to\infty} \sum_{m=n}^{\infty} \mu(E_m) = 0.$$

Thus, we have the desired result. □

Problem 144 (Royden 18.2.1)**.** Prove the following statements:

$$\int_X \alpha g d\mu = \alpha \int_X g d\mu,$$

$$\text{if } g \leq h \text{ a.e. on X, then } \int_X g d\mu \leq \int_X h d\mu,$$

$$\int_X g d\mu = \int_{X_0} g d\mu \text{ if } \mu(X - X_0) = 0.$$

Proof. Recall that

$$\int \alpha \cdot g = \sup \left\{ \int \psi \ : \ \psi \subset SF^+, \ 0 \le \psi \le \alpha \cdot g \right\}.$$

However, we could also write this as

$$\int \alpha \cdot g = \sup \left\{ \int \alpha\psi \ : \ \psi \subset SF^+, \ 0 \le \psi \le \cdot g \right\},$$

and we clearly see that $\int \alpha\psi = \alpha \int \psi$, since

$$\int \alpha\psi = \sum \alpha c_i \mu(E_i) = \alpha \sum c_i \mu(E_i) = \alpha \int \psi.$$

Hence,

$$\int \alpha g = \sup \left\{ \int \alpha\psi \ : \ \psi \subset SF^+, \ 0 \le \psi \le \cdot g \right\} = \alpha \sup \left\{ \int \psi \ : \ \psi \subset SF^+, \ 0 \le \psi \le \cdot g \right\}$$

$$= \alpha \int g.$$

For the next, we clearly have that if $0 \le \psi \le g$, then $0 \le \psi \le h$, and so we get $\int g \le \int h$.

Finally, notice that

$$\int g d\mu = \int g(\chi_{X_0} + \chi_{X_0^c}) d\mu = \int g\chi_{X_0} + \int g\chi_{X_0^c} = \int_{X_0} g d\mu,$$

since

$$\int g\chi_{X_0^c} \le ||g||_\infty \mu(X_0^c) = 0.$$

□

Problem 145 (Royden 20.1.2)**.** Let $\mathbb{N}$ be the set of natural numbers, $\mathcal{M} = \mathcal{P}(\mathbb{N})$, and c the counting measure. Prove that every function $f : \mathcal{N} \to \mathbb{R}$ is measurable with respect to c and that f is integrable over $\mathbb{N}$ with respect to c if and only if the series $\sum_{k=1}^{\infty} f(k)$ is absolutely convergent.

Proof. It's clear that every f is measurable, since the inverse image will be a set, and all sets are measurable in this σ-algebra.

($\Longrightarrow$) Assume f is integrable. We have

$$\int_{\mathbb{N}} |f(x)| dc(x) < \infty.$$

Notice that $\mathbb{N} = \bigsqcup_{n=1}^{\infty} \{n\}$, so we have

$$\int_{\mathbb{N}} |f(x)| dc(x) = \sum_{n=1}^{\infty} \int_{\{n\}} |f(x)| dc(x) = \sum_{n=1}^{\infty} |f(n)| < \infty,$$

so the series is absolutely convergent.

($\Longleftarrow$) If the series is absolutely convergent, then by the relation above, we have that the function is integrable. □

Problem 146 (Folland 4.1)**.** If $\text{Card}(X) \ge 2$, there is a topology on X that is T_0 but not T_1.

Proof. Recall that T_0 states that if $x \ne y$, there is an open set containing x but not y or an open set containing y but not x. Recall that T_1 is if $x \ne y$, there is an open set containing y but not x. Take the topology which is all but one element; i.e. if $X = \{a, b\}$, take the topology $\tau = \{\varnothing, \{a\}, \{a, b\}\}$. Then we see this is T_0 but not T_1. We extend this in the obvious way. □

Problem 147 (Folland 4.2)**.** If X is an infinite set, the cofinite topology on X is T_1 but not T_2, and it is first countable if X is countable.

Proof. Recall that a space is T_2 if for $x \neq y$, there are disjoint open sets U, V such that $x \in U$ and $y \in V$. A space is first countable if there is a countable neighborhood base for every x. The cofinite topology on X is

$$\tau = \{U \subset X \ : \ U = \varnothing \text{ or } U^c \text{ is finite}\}.$$

We check first that this is a topology. Notice that $\varnothing \in \tau$, $X \in \tau$ since $X^c = \varnothing$ is finite. Next, if $\{U_\alpha\}$ is a collection of sets in the topology, we need to show that $\bigcup U_\alpha$ is in the topology as well. Notice that

$$\left(\bigcup U_\alpha\right)^c = \bigcap U_\alpha^c.$$

If U_α is finite for some α, we have that this is finite, and so it is in τ. If they are all the empty set, it is clearly in τ. Hence, it is closed under arbitrary union. Finally, we need to check it is closed under finite intersection. Let $\{U_\alpha\}_{\alpha=1}^n$ be a finite collection of sets in τ. Then we have that $\bigcap_{\alpha=1}^n U_\alpha$ is either the empty set (if one of them is), or we have that

$$\left(\bigcap_{\alpha=1}^n U_\alpha\right)^c = \bigcup_{\alpha=1}^n U_\alpha^c.$$

A finite union of finite things is finite, and so this is in τ. Hence, it's closed under finite intersection, and so it is a topology.

Next, we need to show that it is T_1. Take $x \neq y$. Notice that the set $U_x = \{t \in X \ : \ t \neq x\}$ is in the topology, since the complement is x, and $y \in U_x$. We argue by symmetry that U_y is the same situation. We see it is not T_2; assume U was an open set containing x, V an open set containing y. For them to be disjoint, we need $U \cap V = \varnothing$, but this implies that $U^c \cup V^c = X$, which is infinite, contradicting that U^c and V^c are finite. So this is impossible.

Next, we show it is first countable. That is, we have a countable base at every point x. Since X is countable, enumerate the values $\{x_n \ : \ n \in \mathbb{N}\}$, with $x_0 = x$. Then we can define a neighborhood base via

$$B = \left\{E_n \subset X \ : \ E = X - \bigcup_{i=1}^n \{x_i\}\right\}.$$

These sets are clearly open, since they will always be finite, and they all contain x. Furthermore, we get that it's a base by noticing that if $x \in U$, U open, then we have that it must be missing a finite number of values $\{x_j\}$, and so we can take N sufficiently large so that these are excluded, and hence $E_N \subset U$. $\square$

Problem 148 (Folland 4.3)**.** Every metric space is normal.

Proof. Recall that a space is normal if it is T_4; that is, X is a T_1 space, and for any disjoint closed sets A, B in X, there are disjoint open sets U, V with $A \subset U$ and $B \subset V$. A metric space is T_2 (Hausdorff), and so it is T_1. Next, let A, B be disjoint closed sets in X. We first prove the following claim.

Claim. If D is a closed subset of X, $x \in X$, we have $d(x, D) = 0$ if and only if $x \in D$.

Proof. ($\Longrightarrow$) If $d(x, D) = 0$, then we can construct a sequence x_n of points in D such that $d(x, x_n) < 1/n$. Hence, we have that $x \in \overline{D} = D$.
($\Longleftarrow$) If $x \in D$, then $d(x, D) = 0$. $\square$

Now, for each $x \in A$, let $U_x := B(x, r)$, where we take $r = d(x, B)/3$. Likewise, for each $x \in B$, let $V_x := B(x, r)$, where we take $r = d(x, A)/3$. Let $U = \bigcup_{x \in A} U_x$, $V = \bigcup_{x \in B} V_x$. U and V are open, and we clearly have $A \subset U$, $B \subset V$. We then need to check that $U \cap V = \varnothing$. Let $z \in U \cap V$.

Then we have that $z \in U$, which implies that there is an $x \in A$ such that $d(x,z) < d(x,B)/3 =: r$, and a $y \in B$ such that $d(y,z) < d(y,A)/3 =: s$. Now, without loss of generality take $r \geq s$. Then we have that

$$3r = d(x,B) \leq d(x,y) \leq d(x,z) + d(y,z) \leq r + s \leq 2r.$$

This can only happen if $r = 0$, but if $r = 0$ this implies $z \in A$. A symmetric argument gives $z \in B$ as well, which is impossible. □

Problem 149 (Folland 4.4)**.** Let $X = \mathbb{R}$ and let τ be the family of all subsets of $\mathbb{R}$ of the form $U \cup (V \cap \mathbb{Q})$, where U and V are open in the usual sense. Then τ is a topology that is Hausdorff, but not regular.

Proof. First, recall that regular implies that it is T_3; that is, X is a T_1 space, and for any closed set $A \subset X$ and any $x \in A^c$, there are disjoint open sets U, V with $x \in U$ and $A \subset V$.

We first show that τ is a topology. First, $X \in \tau$, since $X = X \cup (\varnothing \cap \mathbb{Q}) = X$. We have $\varnothing \in \tau$ as well. Let $\{E_\alpha\}$ be an arbitrary collection of sets in τ, then

$$\bigcup E_\alpha = \bigcup (U_\alpha \cup (V_\alpha \cap \mathbb{Q})) = \bigcup U_\alpha \cup \left(\bigcup (V_\alpha) \cap \mathbb{Q} \right) \in \tau.$$

Finally, we have that if $\{E_\alpha\}_{\alpha=1}^n$ is a finite collection, then

$$\bigcap E_\alpha = \bigcap (U_\alpha \cup (V_\alpha \cap \mathbb{Q})) = \bigcap U_\alpha \cup \left(\bigcap V_\alpha \cap \mathbb{Q} \right) \in \tau.$$

Hence, it's a topology.

Next, we check that it is Hausdorff. That is, for $x \neq y$, we can find open sets U, V such that $U \cap V = \varnothing$. Since X is Hausdorff under the usual topology, find U_x, V_x, U_y, V_y all of the differing letters are disjoint. Then it's clear that $x \in U_x, V_x$, $y \in U_y, V_y$, and

$$[U_x \cup (V_x \cap \mathbb{Q})] \cap [U_y \cup (V_y \cap \mathbb{Q})] = (U_x \cap U_y) \cup (U_x \cap (V_y \cap \mathbb{Q})) \cup ((V_x \cap \mathbb{Q}) \cap U_y) \cup ((V_x \cap \mathbb{Q}) \cap (V_y \cap \mathbb{Q})) = \varnothing.$$

Finally, we see it's not T_3. □

Problem 150 (Folland 4.5)**.** Every separable metric space is second countable.

Proof. Let (X, ρ) be our metric space, let A be a countable dense subset of X. We wish to show that X is second countable; that is, there exists a countable base for X. Define the base as

$$\mathcal{B} = \left\{ B_{1/n}(x) \ : \ x \in A, n \in \mathbb{N} \right\}.$$

We see that $\mathcal{B}$ is countable, and so it suffices to show that this is a base. If we can show that every $x \in X$ is in $\bigcup_{U \in \mathcal{B}} U$, we win. Let $y \in X$ arbitrary. By the density property of A, we get that, for every open neighborhood of y, $A \cap \{U - \{y\}\} \neq \varnothing$. The open balls are the base of the topology of (X, ρ), so it suffices to work with $B_\epsilon(y)$ for our open sets. We can find an n sufficiently large so that $2/n < \epsilon$, hence $B_{2/n}(y) \subset B_\epsilon(y)$. Notice that we have that $A \cap (B_{1/n}(y) - \{y\}) \neq \varnothing$. Take x in this intersection. Examine the open set $B_{1/n}(x)$. Take z in this. We have that

$$\rho(z, y) \leq \rho(z, x) + \rho(x, y) < 2/n,$$

so $B_{1/n}(x) \subset B_{2/n}(y) \subset B_\epsilon(y)$. Since we can do this for all open balls, we win. □

Problem 151 (Folland 4.7)**.** If X is a topological space, a point x is called a cluster point of a sequence (x_j) if, for every neighborhood U of x, $x_j \in U$ for infinitely many j. If X is first countable, x is a cluster point of (x_j) if and only if some subsequence of (x_j) converges to x.

Proof. X is first countable, so we have a countable neighborhood basis at x, call it $\mathcal{N}_x$. Assume x is a cluster point of (x_j). Then for every neighborhood U of x, $x_j \in U$ for infinitely many j. Since $\mathcal{N}_x$ is countable, we can define a subsequence as follows: Let U_k be a nested countable neighborhood basis at x; i.e. set $U_k = \bigcap_{i=1}^k N_i$, $N_i \in \mathcal{N}_x$. We can choose x_{n_k} such that $n_k \geq n_{k-1}$ and $x_{n_k} \in U_k$.

Since this is a neighborhood basis, we have that for every neighborhood V of x, there is a U_k such that $U_k \subset V$, and so x_{n_k} is eventually in V. Hence, x_{n_k} converges to x.

For the other direction, it's clear that if a subsequence converges to x, then x_n is in every neighborhood of U infinitely often. □

Problem 152 (Folland 4.30)**.** If A is a directed set, a subset B of A is called **cofinal** in A if for each $\alpha \in A$, there exists a $\beta \in B$ such that $\beta \geq \alpha$.

(1) If B is cofinal in A and $\langle x_\alpha \rangle$ is a net, the inclusion map $B \hookrightarrow A$ makes $\langle x_\beta \rangle$ a subnet of $\langle x_\alpha \rangle$.
(2) If $\langle x_\alpha \rangle$ is a net in a topological space, then $\langle x_\alpha \rangle$ converges to x if and only if for every cofinal $B \subset A$, there is a cofinal $C \subset B$ such that $\langle x_\gamma \rangle$ converges to x.

Proof. (1) Recall a subnet is a net $\langle y_\beta \rangle$ with a map $f : B \to A$ such that for every $\alpha_0 \in A$, there is a $\beta_0 \in B$ such that $\alpha_\beta \geq \alpha_0$ whenever $\beta \geq \beta_0$, and $y_\beta = x_{\alpha_\beta}$. We want to check that $\langle x_\beta \rangle$ is a subnet of $\langle x_\alpha \rangle$. We check the first property: take $\alpha_0 \in A$. Then since B is cofinal, there is a β such that $\beta \geq \alpha$. Take $\beta_0 = \beta$ (this beta); then for all $\beta \geq \beta_0$, we have that $f(\beta) = \beta \geq \alpha$. Furthermore, it's clear that $x_\beta = x_\beta$, and so it's a subnet.
(2) ($\implies$) Let V be a neighborhood of x. Then $\langle x_\alpha \rangle$ converges to x implies that $\langle x_\alpha \rangle$ is eventually in V. Let $B \subset A$ be cofinal. Then it's clear that taking $C = B$ gives a net which converges to x, and so the result finals.
($\impliedby$)

□

Problem 153. $f : X \to Y$ is continuous if and only if for all convergent nets $x_i \to x$ in X, $f(x_i) \to f(x)$ in Y.

Proof. ($\implies$) Assume f is continuous. Let $x_i \to x$ be a convergent net. Then for all open V such that $x \in V$, we have that x_i is eventually in V. Let U be an open neighborhood of $f(x)$. Then $f^{-1}(U)$ is open, since f continuous, and $x \in f^{-1}(U)$, so x_i is eventually in $f^{-1}(U)$, and so $f(x_i)$ is eventually in U. Since this applies for all neighborhoods, $f(x_i)$ converges to $f(x)$.
($\impliedby$) We wish to show that for all closed $A \subset Y$, $B := f^{-1}(A)$ is closed. That is, $\overline{B} = B$. A set is closed if and only if it contains the limits of all convergent nets in it. So take $x_i \to x$, $x_i \in B$, then $x \in \overline{B}$. However, notice that $f(B) \subset A$, and so $f(x_i) \to f(x) \in A$. So $x \in B$. Since we can do this for all $x \in \overline{B}$, we have that $B = \overline{B}$. So B is closed.
Folland proof Assume f not continuous at x for contradiction. Then we have that there is a neighborhood V of $f(x)$ such that $f^{-1}(V)$ is not a neighborhood of x, so $x \in \overline{f^{-1}(V^c)}$. We can construct a net in $f^{-1}(V^c)$ which converges to x, since it's closed, but this gives us the contradiction, since this implies that $f(x_i) \not\to f(x)$. □

Problem 154. Prove that second countable implies separable.

Proof. Second countable means that there is a countable base for the topology. Separable means there is a countable dense subset. Let B be the base. For each $U \in B$, choose an $x \in U$ and let T be this set. Notice that $(\overline{T})^c = \varnothing$, since every point $x \in X$ is such that an open neighborhood intersects some $y \in T$. Thus, $\overline{T} = X$, and T is countable. □

Problem 155. Suppose X is first countable, $A \subset X$. Then $x \in \overline{A}$ if and only if there exists a sequence $x_j \subset A$ such that $x_j \to x$.

Proof. Let $x \in \overline{A}$. Then this implies that x is an accumulation point of A; that is, for every open neighborhood of x, say U, we have $U \cap A \neq \varnothing$. Let $\{B_i\}$ be a countable base at x. Let $U_n = \bigcap_{i=1}^n B_i$. Then we have that $U_{i+1} \subset U_i$ by construction. Notice as well we have $U_i \cap A \neq \varnothing$ for all i, so choose $x_i \in U_i \cap A$. Then for every open neighborhood V of x, we get that there's some n so that $U_n \subset V$, since this is a basis, and so we have that x_n is eventually in V. Hence, x_i converges to x.

On the other hand, take a sequence $x_j \to x$, $(x_j) \subset A$. Let U be a neighborhood of x. Since $x_j \to x$, we have that x_j is in U eventually. Hence, $U \cap A \neq \varnothing$. Since this works for all neighborhoods of x, we get that $x \in \overline{A}$. □

Problem 156. If (X, d) is a metric space, the following are equivalent:

(1) X is compact.
(2) X is sequentially compact.
(3) X is complete and totally bounded.

Proof. (1) $\implies$ (2): Recall that a space is sequentially compact if every sequence of points in X has a convergent subsequence to a point in X. Take $(x_n) \subset X$ a sequence, X compact. Let

$$F_n := \overline{\{x_k \ : \ k \geq n\}},$$
$$U_n := F_n^c.$$

If

$$\bigcap F_n = \varnothing,$$

then we have

$$\bigcup U_n = X,$$

and since X is compact we can form an open subcover;

$$X = \bigcup_{i=1}^{n} U_{n_i}.$$

Notice that this implies that

$$\bigcap_{i=1}^{n} F_{n_i} = \varnothing.$$

However, we have that (x_n) is an infinite sequence, and so this is a contradiction. Hence, we must have

$$x \in \bigcap F_n.$$

Since x is in the closure, this means that we must have that

$$B(x, 1/n) \cap \{x_k \ : \ k \geq n\} \neq \varnothing.$$

Hence, picking x_{n_k} from these, we get a converging subsequence.

(2) $\implies$ (3): Let $\{x_n\}$ be a Cauchy sequence; then x_n has a convergent subsequence, and since limits of Cauchy sequences are unique, this implies that x_n converges. To see this, let x be where the subsequence x_{n_k} converges. Then since it's Cauchy, we have that there is an N such that for all $n, m \geq N$, $d(x_n, x_m) < \epsilon/2$, and since it's convergent there is an N' such that for all $n_k \geq N'$, we have $d(x_{n_k}, x) < \epsilon/2$. Hence, taking N'' to be the max, we have that for all $n \geq N''$,

$$d(x_n, x) \leq d(x_n, x_{N''}) + d(x_{N''}, x) < \epsilon.$$

So $x_n \to x$.

Next, we need to show that X is totally bounded. Recall that totally bounded implies that for $\epsilon > 0$, there is a sequence of points $x_1, \ldots, x_n$ such that $X = \bigcup_{i=1}^{n} B(x_i, \epsilon)$. Assume for contradiction it is not totally bounded. Then there is an ϵ so that there is no collection of elements x_i where $X = \bigcup_{i=1}^{n} B(x_i, \epsilon)$. Choose $x_1 \in X$. Since $X \neq B(x_1, \epsilon)$, there is an $x_2 \in X - B(x_1, \epsilon)$; that is, $d(x_1, x_2) \geq \epsilon$. For $\{x_1, \ldots, x_n\}$, choose x_{n+1} such that $d(x_k, x_{n+1}) > \epsilon$ for all $k \in \{1, \ldots, n\}$. We have a sequence which has no convergent subsequences by construction, contradicting the fact that X is sequentially compact. Hence, X must be totally bounded.

(2) $\implies$ (1): Assume X is sequentially compact. Take a cover $X \subset \bigcup U_i$. Assume that X cannot be covered by a finite refinement. Let $F_n = \bigcup_{i=1}^n U_i$. Then we can choose $x_n \in F_n^c$, and this gives us a sequence. Since X is sequentially compact, it has a subsequence which converges to something, say x. But this implies that there is some m such that $x \in U_m$, so $x_i \in U_m$ for infinitely many values, but this contradicts the choice of the sequence.
(3) $\implies$ (1) : Take an open cover $X = \bigcup U_i$. Assume it cannot be finitely refined. Since X is totally bounded, choose $\epsilon = 1$ to get

$$X = \bigcup_{i=1}^{n} B(x_i, 1).$$

If each of these balls were covered by a finite subcollection of U_i, we win. So assume that $B(x_1, 1)$ is not. Since $B(x_1, 1)$ is still totally bounded, we can cover it with balls with radius 1/2;

$$B(x_1, 1) = \bigcup_{i=2}^{n_1} B(x_i, 1/2).$$

One of these cannot be covered by a finite subcollection of U_i, so choose it to be $B(x_2, 1/2)$. Continue down the line, getting a sequence (x_n) where $d(x_n, x_{n+1}) \leq 2^{-n} + 2^{-n-1}$, and where $B(x_n, 2^{-n})$ is not in any finite collection of U_i. Notice that x_n converges, since it is Cauchy, and so it must converge to some point x. We must have that $x \in U_i$ for some i. Notice that there is an r sufficiently small so that $B(x, r) \subset U_i$, for if not then we have that $x \notin U_i$, a contradiction. By construction, we can find n sufficiently large so that $2^{-n} < r$, but this forces $B(x, 2^{-n}) \subset U$, which contradicts our choice. Since we have a contradiction, we must have that it can be finitely refined. □

Problem 157. Let (X, d) be a complete metric space, $A \subset X$. Prove that $\overline{A}$ is compact if and only if A is totally bounded.

Proof. ($\implies$) If $\overline{A}$ is compact, then it is complete and totally bounded by the theorem from the class notes. For $\epsilon > 0$, take a cover of $\overline{A}$ via

$$\overline{A} \subset \bigcup_{i=1}^{m} B(x_i, \epsilon/2).$$

Assume $B(x_i, \epsilon/2) \cap A \neq \varnothing$ for $i = 1, \dots, n$. Choose $y_i \in B(x_i, \epsilon/2)$. Then we want to show that

$$A \subset \bigcup_{i=1}^{n} B(y_i, \epsilon).$$

Let $y \in A$. Then $y \in B(x_i, \epsilon/2)$. Notice that

$$d(y, y_i) \leq d(y, x_i) + d(x_i, y_i) < \epsilon,$$

so $y \in B(y_i, \epsilon)$. Hence, A is totally bounded.
($\impliedby$) Assume that A is totally bounded. Then we want to show that $\overline{A}$ is also totally bounded. Let $\epsilon > 0$. Then we have that

$$A \subset \bigcup_{i=1}^{n} B(x_i, \epsilon/2).$$

We want to show that

$$\overline{A} \subset \bigcup_{i=1}^{n} B(x_i, \epsilon).$$

Take $y \in \overline{A}$. Then we have that $B(y, \epsilon/2) \cap A \neq \varnothing$. Take x in this intersection; then $d(y,x) < \epsilon/2$. Furthermore, $x \in B(x_i, \epsilon/2)$ for some i, so we have

$$d(y, x_i) \leq d(y,x) + d(x, x_i) < \epsilon.$$

Hence, $y \in B(x_i, \epsilon)$. So $\overline{A}$ is totally bounded.

Next, we want to show that $\overline{A}$ is complete. However, this is clear; let (x_n) be a Cauchy sequence in $\overline{A}$. Then it has a convergent subsequence, and so it must also converge. So every Cauchy sequence converges. □

Problem 158 (Folland 4.10)**.** Prove that, for a connected set, the only open and closed sets are $\varnothing$ and X.

Proof. Recall that a set is disconnected if there exists nonempty open sets U and V such that $U \cup V = X$ and $U \cap V = \varnothing$, and is connected otherwise. Let X be connected, and assume that U is both open and closed. Then we have that U^c is also open and closed, which means that $U \cup U^c = X$, $U \cap U^c = \varnothing$. However, the only sets which satisfy this are $U = \varnothing$ or $U = X$. □

Problem 159 (Folland 4.11)**.** If $E_1, \ldots, E_n$ are subsets of a topological space, the closure of $\bigcup_{i=1}^n E_i$ is $\bigcup_{i=1}^n \overline{E_i}$.

Proof. It suffices to show it for the case of two subsets and then induct. That is, we wish to show that

$$\overline{E_1 \cup E_2} = \overline{E_1} \cup \overline{E_2}.$$

One direction is clear, we have that

$$E_1 \cup E_2 \subset \overline{E_1} \cup \overline{E_2},$$

and the RHS is a closed set, so we must have

$$\overline{E_1 \cup E_2} \subset \overline{E_1} \cup \overline{E_2}.$$

Analogously, we have

$$E_1 \subset E_1 \cup E_2,$$

so

$$\overline{E_1} \subset \overline{E_1 \cup E_2},$$

and likewise

$$\overline{E_2} \subset \overline{E_1 \cup E_2},$$

so

$$\overline{E_1} \cup \overline{E_2} \subset \overline{E_1 \cup E_2}.$$

Hence

$$\overline{E_1} \cup \overline{E_2} = \overline{E_1 \cup E_2},$$

as desired. □

Problem 160 (Folland 4.13)**.** If X a topological space, U is open in X, and A is dense in X, then $\overline{U} = \overline{U \cap A}$.

Proof. First, remark that

$$U \cap A \subset U,$$

and so therefore

$$\overline{U \cap A} \subset \overline{U}.$$

We then wish to show the other direction; that is, $\overline{U} \subset \overline{U \cap A}$. Next, we have that U is open, so U^c is closed. Notice as well that $\overline{U \cap A}$ is closed as well. A finite union of closed sets is closed, so $U^c \cup \overline{U \cap A}$ is closed. Notice that $A \subset U^c \cup \overline{U \cap A}$ (take $x \in A$, either $x \in U \cap A \subset \overline{U \cap A}$ or $x \in U^c \cap A \subset U^c$), and since A is dense, we have that

$$\overline{A} = X \subset U^c \cup \overline{U \cap A} \subset X \implies U^c \cup \overline{U \cap A} = X.$$

Hence, we get

$$U = X \cap U = U \cap \left(U^c \cup \overline{U \cap A}\right) = (U \cap U^c) \cup (U \cap \overline{U \cap A}) = U \cap \overline{U \cap A}.$$

In other words, we have

$$U \subset \overline{U \cap A}.$$

This then gives us

$$\overline{U} \subset \overline{U \cap A}$$

as desired. □

Problem 161 (Folland 4.14)**.** If X and Y are topological space, the following are equialent:

(1) $f : X \to Y$ is continuous.
(2) $f(\overline{A}) \subset \overline{f(A)}$ for all $A \subset X$.
(3) $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$ for all $B \subset Y$.

Proof. (1) $\implies$ (2) : If f is continuous, we have pullback of closed sets are closed. Hence, we have $f^{-1}(\overline{f(A)})$ is closed. Notice that $A \subset f^{-1}(\overline{f(A)})$, and so $\overline{A} \subset f^{-1}(\overline{f(A)})$ by minimality of the closure. Applying f to both sides gives $f(\overline{A}) \subset \overline{f(A)}$.
(2) $\implies$ (3) : Since it applies for all A, take $B \subset Y$ and notice that $f^{-1}(B) \subset X$. Then $f(\overline{f^{-1}(B)}) \subset \overline{f(f^{-1}(B))} \subset \overline{B}$. Hence, $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$.
(3) $\implies$ (1): Let $B \subset Y$ be a closed set. Then we have

$$\overline{f^{-1}(B)} \subset f^{-1}(B) \subset \overline{f^{-1}(B)} \implies f^{-1}(B) = \overline{f^{-1}(B)}.$$

Hence, it pulls back closed sets to closed sets, which means f is continuous. □

Problem 162 (Folland 4.15)**.** If X is a topological space, $A \subset X$ is closed, and $g \in C(A)$ satisfies $g = 0$ on ∂A, then the extension of g to X defined by $g(x) = 0$ for $x \in A^c$ is continuous.

Proof. We assume the codomain is $\mathbb{C}$ for now. Let $E \subset \mathbb{C}$ be open; we have that $g^{-1}(E) \subset A$ is open with respect to the subspace topology. Letting $\hat{g}$ denote the extension to X, we need to show that $\hat{g}^{-1}(E)$ is open. Let $B = \{x \;:\; \hat{g}(x) \neq 0\} \subset A$, $B^c = \{x \;:\; \hat{g}(x) = 0\}$. Then we have

$$\hat{g}^{-1}(E) = \left(\hat{g}^{-1}(E) \cap B\right) \sqcup \left(\hat{g}^{-1}(E) \cap B^c\right).$$

Since B is open, we get that the left hand side is open. We then wish to establish that the right hand side is open as well. We examine

$$(\hat{g}^{-1}(E) \cap B^c \cap A^{o}) \cup (\hat{g}^{-1}(E) \cap B^c \cap \partial A) \cup (\hat{g}^{-1}(E) \cap A^c).$$

Notice that the left most set is open, the right most set is open (either E contains 0 or it doesn't), so it suffices to check that the middle set is open, but this follows by $g \in C(A)$. □

Problem 163 (Folland 4.16)**.** Let X be a topological space, Y a Hausdorff space, and f, g continuous maps from X to Y.

(1) $\{x \;:\; f(x) = g(x)\}$ is closed.
(2) If $f = g$ on a dense subset of X, then $f = g$ on all of X.

Proof.

(1) Let $(x_n) \subset F := \{x \;:\; f(x) = g(x)\}$ be a convergent net. Then, since f, g are continuous, we have $f(x_n) \to f(x)$, $g(x_n) \to g(x)$. Since $f(x_n) = g(x_n)$ for all n, we must have $f(x) = g(x)$ (since Y is Hausdorff), and so $x \in F$. Since this is true for all convergent nets, we have that F is closed.

(2) This condition tells us that F is dense. Since F is closed, we have that the closure is the whole set, but $\overline{F} = F = X$.

$\square$

Problem 164 (Folland 4.18)**.** If X and Y are topological spaces, $y_0 \in Y$, then X is homeomorphic to $X \times \{y_0\}$, where the latter has the relative topology as a subset of $X \times Y$.

Proof. We wish to construct $f : X \times \{y_0\} \to X$ which is bijective, continuous, and has continuous inverse. First, notice that $f(x, y_0) = x$ is the projection map, which is clearly well-defined and surjective. For injective, notice that $f(x, y_0) = f(x', y_0) \iff x = x'$, and so $(x, y_0) = (x', y_0)$. Finally, we need to show it's continuous. Take $U \subset X$ open, then $f^{-1}(U) = U \times \{y_0\}$. This is an open subspace of $X \times \{y_0\}$, since we can write it as $(U \times Y) \cap (X \times \{y_0\})$. Hence, f is bijective and continuous.

Notice $f^{-1}(x) = g(x) = (x, y_0)$. We need to show that this is continuous as well. Let $U \subset X \times \{y_0\}$ be open, then we have that $U = U' \cap (X \times \{y_0\})$, where U' is open. Notice that U' is open in the product topology, so we have that the projection map is continuous on it; hence, $\pi_X(U')$ is open. This tells us that $g^{-1}(U)$ is open, and so this is a continuous map. Hence, f we have a homeomorphism. $\square$

Problem 165 (Folland 4.32)**.** A topological space X is Hausdorff if and only if every net in X converges to at most one point.

Proof. ($\implies$) Assume X is Hausdorff. Then we need to show that every net in X converges to at most one point. Assume (x_i) is a convergent net in X. This means that $x_i \to x$, which means that for every neighborhood V of x, we have that x_i is eventually in V. Assume for contradiction that $x_i \to y$ as well. Since X Hausdorff, we can find disjoint open neighborhoods U, V such that $x \in U$, $y \in V$. Since x_i is eventually in U, we have that there is an N such that for all $n \geq N$, $x_n \in U$. But this means that (x_i) is not eventually in V, and so we have that x_i cannot converge to y. Hence, there's at most one.
($\impliedby$) Assume X is not Hausdorff. Then there are $x, y \in X$ such that there are no open neighborhoods U, V where $x \in U$, $y \in V$, $U \cap V = \varnothing$. Consider the directed set $\mathcal{N}_x \times \mathcal{N}_y$, where $\mathcal{N}_x$, $\mathcal{N}_y$ are the families of neighborhoods of x, y. Notice we can make this an ordered set in the following way; we have $(U_x, U_y) \geq (V_x, V_y)$ if $V_x \subset U_x$ and $V_y \subset U_y$. Notice that for any $(U_x, U_y), (V_x, V_y)$, we have that there is a (T_x, T_y) such that $(T_x, T_y) \geq (U_x, U_y)$ and $(T_x, T_y) \geq (V_x, V_y)$; if there were not any, we would have that there were open neighborhoods about x and y which were disjoint. We can define a subnet $(x_{N_x,N_y})_{(N_x,N_y) \in \mathcal{N}_x \times \mathcal{N}_y}$ which converges to both x and y. $\square$

Problem 166 (Folland 4.38)**.** Suppose that (X, τ) is a compact Hausdorff space, and τ' is another topology on X. If τ' is strictly stronger than τ (read: contains more elements, i.e. $\tau \subset \tau'$), then (X, τ) is Hausdorff but not compact. If τ' is strictly weaker than τ, then (X, τ') is compact but not Hausdorff.

Proof. If τ' is stronger, then we clearly see that for every $x, y \in X$, we can find open U_x, U_y so that $U_x \cap U_y = \varnothing$ and the elements are in their respective sets. Take $f : (X, \tau') \to (X, \tau)$ defined by $f(x) = x$ (the identity). Then for all $U \subset \tau$, we have $f^{-1}(U) = U \in \tau'$, since τ' is stronger, so f is continuous. Furthermore, it is a bijection. However, it is not a homeomorphism, since there are $U \in \tau'$ such that $f(U) = U \notin \tau$. By the contrapositive of **Proposition 4.28**, we get that since X is Hausdorff, X must not be compact under the stronger topology.

Next, assume τ' is weaker. Then by the same argument, we get that $f : (X, \tau) \to (X, \tau')$ defined by the identity is continuous, bijective, and so $f(X) = X$ is compact by **Proposition 4.26**. However, it is not a homeomorphism by the same argument above, and so we have that (X, τ') must not be Hausdorff. $\square$

Problem 167 (Folland 4.40)**.** If X is countably compact, then every sequence in X has a cluster point. If X is also first countable, then X is sequentially compact.

Proof. Let (x_n) be a sequence in X. Let $E_k = \bigcup_{n=k}^{\infty}\{x_n\}$. If $\bigcap_{k=1}^{\infty}\overline{E_k} = \varnothing$, then we have that $\bigcup_{k=1}^{\infty}\overline{E_k}^c = X$, and so we can take a finite refinement to get $\bigcup_{j=1}^{N}\overline{E_{k_j}}^c = X$, which implies that $\bigcap_{j=1}^{N}\overline{E_{k_j}} = \varnothing$. This tells us that the sequence terminates after a finite number of steps, contradicting the fact that it's a sequence. Hence, we must have that $\bigcap_{k=1}^{\infty}\overline{E_k} \neq \varnothing$. Take x in this intersection. Let V be a neighborhood of x. We have that $(V - \{x_0\}) \cap E_k \neq \varnothing$ for some k, so for all $n \geq k$, we have that $x_k \in V$. This tells us that x is a cluster point of (x_k).

If X is first countable, by **Folland 4.7**, we have that every sequence has a cluster point if and only if every sequence has a convergent subsequence. Since every sequence has a cluster point, we get that every sequence has a convergent subsequence, and so it is sequentially compact. ☐

Problem 168 (Folland 4.44)**.** If X is countably compact and $f : X \to Y$ continuous, then $f(X)$ is countably compact.

Proof. Let $\{V_\alpha\}$ be an open cover of $f(X)$. Then we have

$$f(X) \subset \bigcup_{\alpha} V_\alpha.$$

Notice that this implies

$$X \subset \bigcup_{\alpha} f^{-1}(V_\alpha).$$

Since X compact, we have

$$X \subset \bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i}).$$

Taking the image gives

$$f(X) \subset \bigcup_{i=1}^{n} V_{\alpha_i}.$$

Hence, $f(X)$ is compact. ☐

Problem 169 (Folland 4.45)**.** If X is normal, then X is countably compact iff $C(X) = BC(X)$.

Proof. Recall a space is normal if it is T_1 and for any disjoint closed sets A, B in X, there are disjoint open sets U, V with $A \subset U$ and $B \subset V$. Recall as well that $C(X) = \{f : X \to F \ : \ f \text{ is continuous}\}$, $BC(X) = \{f : X \to F \ : \ f \text{ is bounded and continuous.}\}$.

Assume X is normal and countably compact. Let $f \in C(X)$. Notice that $f(X)$ is compact by the prior problem, and compact in $F = \mathbb{R}$ or $\mathbb{C}$ implies closed and bounded. Hence, f is bounded, and so we have $BC(X) = C(X)$.

Assume X is not compact. Let $\{V_\alpha\}$ be a cover of X which does not admit a finite refinement. Define a sequence via selecting $x_n \in (\bigcup_{\alpha=1}^{n} V_\alpha)^c$. We want to show that (x_n) has no accumulation points. If $x \in X$, we have that $x \in V_n$ for some n. So taking a neighborhood $U \subset V_n$ of x, we have that for N sufficiently large, $x_k \notin U$ for all $k \geq N$. Hence, x is not cluster point of (x_n). Since the set of cluster points is empty, we have that $C := \{x_n\}$ is closed. Define $f : C \to \mathbb{R}$ by $f(x) = \max\{n \ : \ x = x_n\}$. We see that this is unbounded, and furthermore we use **Corollary 4.17** to find $F \in C(X)$ such that $F|_C = f$. so $C(X) \neq BC(X)$. ☐

Problem 170 (Folland 4.47)**.** If X, X^* are such that $X^* = X \cup \{\infty\}$, and τ is the collection of all subsets of X^* such that either U is an open subset of X or $\infty \in U$ and U^c is a compact subset of X, then (X^*, τ) is a compact Hausdorff space, and the inclusion $i : X \to X^*$ is an embedding.

Moreover, if $f \in C(X)$, then f extends continuously to X^* if and only if $f = g+c$, where $g \in C_0(X)$ and c is a constant, in which case the continuous extension is given by $f(\infty) = c$.

Proof. We break this up into parts.

(1) We first show that (X^*, τ) is compact. Let $\{V_\alpha\}$ be an open cover of X^*; that is,

$$X^* \subset \bigcup_\alpha V_\alpha.$$

Since V_α are open and cover X^*, there must be a V_β such that $\infty \in V_\beta$ and V_β^c is compact. Since

$$X^* = V_\beta \cup V_\beta^c,$$

we get

$$V_\beta \cup V_\beta^c \subset \bigcup_\alpha V_\alpha \cup V_\beta.$$

Notice that this means

$$V_\beta^c \subset \bigcup_\alpha (V_\alpha \cap V_\beta^c),$$

and so taking a finite refinement we have

$$V_\beta^c \subset \bigcup_{i=1}^n (V_{\alpha_i} \cap V_\beta^c),$$

hence

$$X^* = V_\beta \cup V_\beta^c \subset \bigcup_{i=1}^n V_{\alpha_i} \cup V_\beta.$$

So we have that there is a finite refinement, and so it is compact.

(2) We now show it is Hausdorff. Take $x, y \in X^*$. If $x \neq y$ and both are not infinite, we have that we can use the Hausdorff property from X to find the open sets. Suppose $y = \infty$, $x \neq y$. Since X LCH, find compact neighborhood U of x. Then U^c is an open neighborhood containing ∞ such that $U^c \cap U = \varnothing$. So we win.

(3) We show that $i : X \to X^*$ is an embedding. Clearly, it's injective. We see it's also a homeomorphism onto it's image, since $i(U) = U$ is open in τ, and the only open sets in $i(X)$ are the ones already open in X. So, it's an embedding.

(4) Omitted.

□

Problem 171 (Folland 4.49)**.** Let X be a compact Hausdorff space and $E \subset X$.

(1) If E is open, then E is locally compact in the relative topology.
(2) If E is dense in X and locally compact in the relative topology, then E is open.
(3) E is locally compact in the relative topology iff E is relatively open in $\overline{E}$.

Proof.

(1) Take $x \in E$. By **Proposition 4.30**, since $x \in E$, E open, we have that there is a compact neighborhood N of x such that $N \subset E$. We can then find an open neighborhood U of x so that $U \subset N$. Now $U \cap E$ open in the relative topology of E, so we get that N is a compact neighborhood of x in the relative topology as well.

(2) If, for all $x \in E$, we can find an open U such that $x \in U \subset E$, then we have that E is open. Take $x \in E$ arbitrarily, then. Since E LCH in the relative topology, we get that we can find a compact neighborhood N such that $x \in N \subset E$. Thus, we can find a relatively

open neighborhood U of x such that $x \in U \subset N$. Since U is relatively open, we have that $U = V \cap E$, where V is open. Since E open, we have that $\overline{V} = \overline{V \cap E}$. Hence, we have

$$x \in V \subset \overline{V} = \overline{V \cap E} \subset N \subset E.$$

Thus, we win.

(3) Omitted

□

Problem 172 (Folland 4.50)**.** Let U be an open subset of a compact Hausdorff space X and U^* its one-point compactification. If $\phi : X \to U^*$ is defined by $\phi(x) = x$ if $x \in U$ and $\phi(x) = \infty$ if $x \in U^c$, then ϕ is continuous.

Proof. Let $V \subset U^*$ be open. There are two cases to consider: either $\infty \in V$ or not. If $\infty \notin V$, we have that $V \subset U$ is open, and so $\phi^{-1}(V) = V \subset U$ open. Since U open, we get that V open in X. Now, if $\infty \in V$, we have that $V^c \subset U \subset X$ is compact. Since X Hausdorff, we get that V^c is closed, so $\phi^{-1}(V^c) = \phi^{-1}(V)^c$ is closed. This gives us that $\phi^{-1}(V)$ is open, as desired. □

Problem 173 (Folland 5.2)**.** We have $\mathcal{L}(X,Y) = \{T : X \to Y \ : \ T \text{ is bounded}\}$ is a vector space, and the function $||\cdot||$ defined by

$$||T|| = \sup\{||Tx|| \ : \ ||x|| = 1\}$$

is a norm. Moreover, prove that

$$\sup\{||Tx|| \ : \ ||x|| = 1\} = \sup\left\{\frac{||Tx||}{||x||} \ : \ ||x|| \neq 0\right\} = \inf\{C \ : \ ||Tx|| \leq C||x|| \text{ for all } x\}$$

Proof. It's clear that $\mathcal{L}(X,Y)$ is a vector space; we have that for all $f, g \in \mathcal{L}(X,Y)$, $f-g$ is linear and bounded, (since $||(f-g)(x)|| = ||f(x) - g(x)|| \leq ||f(x)|| + ||g(x)|| \leq C||x|| + D||x|| = (C+D)||x||$), and it's also closed under scaling ($||cf(x)|| \leq |c|||f(x)|| \leq |c|C||x||$). The other properties are clear after similar calculations.

To see it's a norm, we need to show three things.

(1) Notice that

$$||cT|| = \sup\{||cT(x)|| \ : \ ||x|| = 1\} = \sup\{|c| \cdot ||T(x)|| \ : \ ||x|| = 1\} = |c| \sup\{||T(x)|| \ : \ ||x|| = 1\}$$
$$= |c| \cdot ||T||.$$

(2) We have that

$$||T+S|| = \sup\{||(T+S)(x)|| \ : \ ||x|| = 1\} = \sup\{||T(x) + S(x)|| \ : \ ||x|| = 1\}$$
$$\leq \sup\{||T(x)|| + ||S(x)|| \ : \ ||x|| = 1\} \leq ||T|| + ||S||.$$

(3) Finally, let $||T|| = 0$. Then we have

$$\sup\{||T(x)|| \ : \ ||x|| = 1\} = 0.$$

Since $0 \leq ||T(x)||$ for all $x \in X$, this gives

$$0 \leq ||T(x)|| \leq 0 \implies ||T(x)|| = 0.$$

for all x such that $||x|| = 1$. Hence, $T(x) = 0$ for all x such that $||x|| = 1$. By scaling, we see that this gives us $T(x) = 0$ for all x, and so $T = 0$.

Hence, we have that this is a norm.

Finally, we check that these sets are equal. Notice that

$$\frac{||T(x)||}{||x||} = \left|\left|T\left(\frac{x}{||x||}\right)\right|\right|,$$

and norm of $x/||x||$ is 1. Hence, the first two values are clearly equal. Let C be the infimum, r the supremum. Then we have

$$||T(x)|| \leq C||x|| \implies \frac{||T(x)||}{||x||} \leq C,$$

so taking the sup of the left hand side yields that $r \leq C$. For the other direction, for all x, we have

$$\frac{||T(x)||}{||x||} \leq r \implies ||T(x)|| \leq r||x||,$$

and so $C \leq r$. Hence, $r = C$. □

Problem 174 (Folland 5.3)**.** Complete the proof of **Proposition 5.4**. That is, prove that if Y is complete, then so is $\mathcal{L}(X, Y)$.

Proof. Let (T_n) be a Cauchy sequence in $\mathcal{L}(X, Y)$. If $x \in X$, then $(T_n(x))$ is Cauchy in Y, since $||T_n(x) - T_m(x)|| \leq ||T_n - T_m||||x||$. Define $T : X \to Y$ by $T(x) = \lim_n T_n(x)$. We need to show that $T \in \mathcal{L}(X, Y)$; that is, it's linear and bounded. To see linear, notice that

$$T(x-y) = \lim_n T_n(x-y) = \lim_n T_n(x) - \lim_n T_n(y) = T(x) - T(y),$$

and for α in the underlying field,

$$T(\alpha x) = \lim_n T(\alpha x) = \alpha \lim_n T_n(x) = \alpha T(x).$$

To see it's bounded, take $x \in X$. Then we have

$$||T(x)|| = ||\lim_n T_n(x)|| \leq \lim_n C_n ||x|| = C||x||,$$

since $(||T_n||) = (C_n)$ is Cauchy as real numbers (to see this: notice that the reverse triangle inequality gives $|\|T_n\| - \|T_m\|| \leq ||T_n - T_m||$). Finally, we need to show that $T_n \to T$. Take $x \in X$ arbitrary. Let $\epsilon > 0$, then since (T_n) Cauchy we have that there is an N such that for all $n, m \geq N$, we have

$$||T_n(x) - T(x)|| = ||(T_n - T)(x)|| = \lim_{m \to \infty} ||(T_n - T_m)(x)|| < \epsilon ||x||,$$

so letting $\epsilon \to 0$ gives $||T_n - T|| \to 0$. To get the in fact, notice that $|||T_n|| - ||T||| \leq ||T_n - T||$, so $||T_n|| \to ||T||$. □

Problem 175 (Folland 5.4)**.** If X, Y are normed vector spaces, the map $(T, x) \mapsto Tx$ is continuous from $\mathcal{L}(X, Y) \times X$ to Y.

Proof. We need to show that if $T_n \to T$, $x_n \to x$, then $T_n(x_n) \to T(x)$. Notice that we can write

$$||T_n(x_n) - T(x)|| = ||T_n(x_n) - T_n(x) + T_n(x) - T(x)|| \leq ||T_n(x_n) - T_n(x)|| + ||T_n(x) - T(x)||.$$

On the left, we have

$$||T_n(x_n - x)|| \leq ||T_n||||x_n - x|| \to 0.$$

On the right, we have

$$||T_n(x) - T(x)|| \leq ||T_n - T||||x|| \to 0.$$

□

Problem 176 (Folland 5.12 b)**.** Let X be a normed vector space and M a proper closed subspace of X. Let $||x + M|| = \inf\{||x + y|| \ : \ y \in M\}$ be the norm on X/M. For any $\epsilon > 0$, there exists $x \in X$ such that $||x|| = 1$ and $||x + M|| \geq 1 - \epsilon$.

Proof. Since M proper, we have that $X - M$ is non-empty. Hence, take $x \in X - M$. If $\epsilon \geq 1$, then we are done, simply by normalizing $x/||x||$. Otherwise, we have $0 < \epsilon < 1$, and so $0 < 1 - \epsilon < 1$, and hence $1 < 1/(1-\epsilon)$, so $||x + M|| < (||x + M||)/(1-\epsilon)$. Since this is an infimum, we have that there is a $y \in M$ such that $||x + y|| < (||x + M||)/(1-\epsilon)$. Let $z = (||x + y||)^{-1}(x + y)$. Then $||z|| = 1$, and furthermore we have

$$||z + M|| = \frac{||(x + y) + M||}{||x + y||} = \frac{||x + M||}{||x + y||} > 1 - \epsilon.$$

□

Problem 177 (Folland 5.12 c)**.** Given the conditions of the last problem, prove that the canonical projection map $\pi : X \to X/M$ has norm 1.

Proof. Recall that

$$||\pi|| = \sup\{||\pi(x)|| \ : \ ||x|| = 1\}.$$

Writing this out, we get

$$||\pi|| = \sup\{||x + M|| \ : \ ||x|| = 1\}.$$

It's clear that $||\pi|| \leq 1$. It suffices, then, to show that $||\pi|| \geq 1$. To see this, we use the prior problem, taking the infimum over all ϵ. □

Problem 178 (Folland 5.15)**.** Suppose that X and Y are normed vector spaces and $T \in \mathcal{L}(X, Y)$.

(1) Show that $\ker(T)$ is a closed subspace.
(2) There is a unique $S \in \mathcal{L}(X/\ker(T), Y)$ such that $T = S \circ \pi$ where π is the canonical projection. Moreover, $||S|| = ||T||$.

Proof. (1) Note that T is continuous. Let $x_n \in \ker(T)$, and suppose $x_n \to x$. Then we need to show that $x \in T$. But this is clear, since

$$T(x) = T(\lim x_n) = \lim T(x_n) = \lim 0 = 0,$$

so $x \in \ker(T)$.

(2) The uniqueness is clear from the universal property of quotients. Notice that $||T|| \leq ||S||$, and $||S|| \leq ||T||$ is a simple calculation which I'll omit.

□

Problem 179 (Folland 5.17)**.** A linear functional f on a normed vector space X is bounded iff $f^{-1}(\{0\})$ is closed.

Proof. ($\implies$) If f is bounded, then f is continuous, and so $f^{-1}(\{0\})$ is closed.
($\impliedby$) If $f^{-1}(\{0\})$ is closed, then this means that $\ker(f)$ is closed in X. If $\ker(f) = X$, then this means that f is the zero map, and so is clearly bounded. Hence, assume its proper. By 5.12(b), we see that there exists a $x \in X$ such that $||x|| = 1$ and $||x + \ker(f)|| \geq \frac{1}{2}$. In particular, for all $y \in \ker(f)$, we have $||x + y|| \geq \frac{1}{2}$. So, if y is such that $||y|| \leq 1/2$, we have $x + y \notin \ker(f)$. Notice that $\{f(y) \ : \ ||y|| \leq 1/2\}$ is connected and symmetric about zero, so it is either bounded or it is all of $\mathbb{R}$. If it is all of $\mathbb{R}$, then we have $f(x + y) \in \mathbb{R}$, so $f(x + y) = f(x) + f(y) = 0$, but this means that $x + y \in \ker(f)$ for $||y|| \leq 1/2$, contradicting what we noted earlier. □

Problem 180 (Folland 5.18)**.** Let X be a normed vector space.

(1) If M is a closed subspace and $x \in X - M$, then $M + \mathbb{C}x$ is closed.
(2) Every finite-dimensional subspace of X is closed.

Proof. (1) Since M is closed, $x \notin M$, we have that there is a $f \in X^* = \mathcal{L}(X, \mathbb{C})$ such that $f(x) \neq 0$ and $f|_M = 0$. In particular, $||f|| = 1$ and $f(x) = \inf_{y \in M} ||x - y||$. We want to show that $M + \mathbb{C}x$ is closed, so take a sequence $(y_n + \alpha_n x) \subset M + \mathbb{C}x$, and suppose it converges to $z \in X$. Then we wish to show that $z \in M + \mathbb{C}x$. Notice that

$$f(z) = f\left(\lim_{n\to\infty}(y_n + \alpha_n x)\right) = \lim_{n\to\infty} f(y_n + \alpha_n x) = \lim_{n\to\infty} \alpha_n f(x),$$

so

$$\lim_{n\to\infty} \alpha_n = \frac{f(z)}{f(x)} =: \alpha.$$

Hence,

$$\lim_{n\to\infty} \alpha_n x = \alpha x.$$

Notice that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty}(y_n + \alpha_n x) - \alpha_n x = z - \alpha x,$$

and furthermore notice that $z - \alpha x \in M$ since M is closed. Hence, $z \in M + \mathbb{C}x$, and so $M + \mathbb{C}x$ is closed.

(2) Notice $M = \{0\}$ is clearly closed, and so $M_1 = \{0\} + \mathbb{C}x = \mathbb{C}x$ is closed by (1). Inducting gives every finite dimensional vector space is closed.

□

Problem 181 (Folland 5.19)**.** Let X be an infinite dimensional normed vector space.

(1) There is a sequence (x_j) in X such that $||x_j|| = 1$ and $||x_j - x_k|| \geq 1$ for all $j \neq k$.
(2) X is not locally compact.

Proof. (1) Select $x_1 \in X$ and normalize so that $||x_1|| = 1$. Now, notice that $M_1 := \mathbb{C}x_1$ is a closed vector space by the prior problem. Choose $y \in X - M_1$, let $f_1 := \inf_{z\in M} ||y - z||$. Notice $f_1 > 0$, since $y \notin M_1$. Choose $u \in M_1$ such that $||y - u|| < 2f_1$. Let $x_2 := (y - u)/(||y - u||)$. Then we have $||x_2|| = 1$, and for all $v \in M_1$,

$$||x_2 - v|| = \left|\left|\frac{y-u}{||y-u||} - v\right|\right| = \frac{1}{||y-u||}||y - u - v||y-u|||| > \frac{1}{2f_1}f_1 = \frac{1}{2}.$$

So in particular, notice that $||x_2 - x_1|| \geq 1/2$. Induct using the same procedure.

(2) Scale around things, notice that we can make a sequence with no convergent subsequences.

□

Problem 182 (Folland 5.20)**.** If $\mathcal{M}$ is a finite-dimensional subspace of a normed vector space X, there is a closed subspace $\mathcal{N}$ such that $\mathcal{M} \cap \mathcal{N} = \{0\}$ and $\mathcal{M} + \mathcal{N} = X$.

Proof. Let $e_1, \ldots, e_n$ be a base of $\mathcal{M}$. Consider $T : k^n \to \mathcal{M}$ via $T(a_1, \ldots, a_n) = \sum a_i e_i$. We get that T is bounded (equipping k^n with the norm $||(a_1, \ldots, a_n)|| = \sum |a_i|$); notice that

$$||T(x)|| = \left\|T\left(\sum a_i e_i\right)\right\| = \left\|\sum a_i T(e_i)\right\| \leq M\sum |a_i|,$$

where $M = \max\{\|T(e_i)\|\}$. So T is continuous. Furthermore, the same argument gives us that T^{-1} is also continuous (bounded). Let $\pi_1, \ldots, \pi_n : k^n \to k$ be the canonical projection maps. Then we have that $f_m := T^{-1} \circ \pi_m$ is a linear functional from $\mathcal{M}$ to k. We see that

$$||f_m(x)|| = ||T^{-1} \circ \pi_m(x)|| \leq ||T^{-1}||||x||,$$

and so we can use Hahn-Banach to extend this to $F_1, \ldots, F_n : X \to k$. Taking $\mathcal{N} = \bigcap_{i=1}^n \ker(F_i)$, we see that this is closed (since the kernel is closed), and we see that

$$\mathcal{M} \cap \mathcal{N} = \mathcal{M} \cap \bigcap_{i=1}^{n} \ker(F_i) = \bigcap_{i=1}^{n} \ker(f_i) = \{0\}.$$

Furthermore, $\mathcal{M} + \mathcal{N} = X$. □

Problem 183 (Folland 5.22)**.** Suppose that X and Y are normed vector spaces and $T \in \mathcal{L}(X, Y)$. Define $T^\dagger : Y^* \to X^*$ by $T^\dagger f = f \circ T$. Then $T^\dagger \in \mathcal{L}(Y^*, X^*)$ and $||T^\dagger|| = ||T||$. (This is the adjoint).

Proof. We check linearity first. Notice that $T^\dagger(f+g) = (f+g)\circ T = f\circ T + g \circ T = T^\dagger(f) + T^\dagger(g)$. Notice as well that $T^\dagger(af) = (af)\circ T = a(f \circ T) = aT^\dagger(f)$. Next, pick f such that $||f|| = 1$. Then we have that

$$||T^\dagger f|| = ||f \circ T|| \leq ||f|| ||T|| = ||T||,$$

so it is bounded. Finally, we need to show that $||T|| \leq ||T^\dagger||$. Pick x such that $||x|| = 1$ and $||Tx|| > ||T|| - \epsilon$ (such an x exists by supremum properties). Then Hahn-Banach gives us an f with $||f|| = 1$ and $f(T(x)) = ||T(x)||$. Hence,

$$||T^\dagger(f)(x)|| = ||f \circ T(x)|| = ||T(x)|| > ||T|| - \epsilon.$$

We can do this for all $\epsilon > 0$, so we get that $||T^\dagger|| \geq ||T||$. ☐

Remark. Something worth noting (but not worth doing) is that a closed subspace of a Banach space is Banach, and a subspace of a Banach space which is also Banach is closed.

Problem 184 (Folland 5.35)**.** Let X and Y be Banach spaces, $T \in \mathcal{L}(X, Y)$, $\mathcal{N} = \ker(T)$, $\mathcal{M} = \operatorname{Im}(T)$. Then $X/\mathcal{N}$ is isomorphic to $\mathcal{M}$ iff $\mathcal{M}$ is closed.

Proof. First, notice that $\mathcal{N}$ is closed since T is continuous. Hence, X Banach gives us that $X/\mathcal{N}$ is Banach. Denote by π the canonical quotient map, i.e. $\pi : X \to X/\mathcal{N}$.
($\Longrightarrow$) Suppose $X/\mathcal{N}$ is isomorphic to $\mathcal{M}$. Since $X/\mathcal{N}$ is Banach by the above remark, we get that $\mathcal{M}$ is Banach, which means it's closed.
($\Longleftarrow$) Suppose that $\mathcal{M}$ is closed. We have the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\;T\;} & \mathcal{M} \\ {\scriptstyle\pi}\downarrow & \nearrow_{\overline{T}} & \\ X/\mathcal{N} & & \end{array}$$

where $\overline{T}$ is defined in the canonical way; $\overline{T}(\pi(x)) = \overline{T}(x + \mathcal{N}) = T(x)$. Notice that $\overline{T}$ is linear apriori, and we have that $||\overline{T}|| = ||T||$ by prior exercises, so $\overline{T}$ is bounded. Furthermore, $\overline{T}$ is injective (by construction) and $\overline{T}$ is surjective (by construction), so it is a bounded bijective linear map. The open mapping theorem gives us that this is an isomorphism. ☐

Problem 185 (Folland 5.36)**.** Let X be a separable Banach space and let μ be the counting measure on $\mathbb{N}$. Suppose that (x_n) is a countable dense subset of the unit ball of X, and define $T : \mathcal{L}^1(\mu) \to X$ by $T(f) = \sum_1^\infty f(n)x_n$.

(1) Show that T is bounded.
(2) Show that T is surjective.
(3) Show that X is isomorphic to a quotient space of $\mathcal{L}^1(\mu)$.

Proof. (1) We wish to show that, for all $f \in \mathcal{L}^1(\mu)$, $||T(f)|| \leq M||f||$. Here, the norm on $\mathcal{L}^1(\mu)$ is defined by

$$\|f\| = \int |f| d\mu = \sum |f(n)|.$$

Hence,

$$\|T(f)\| = \left\| \sum_1^\infty f(n) x_n \right\| \leq \sum_1^\infty |f(n)| \|x_n\| \leq \sum_1^\infty |f(n)| = \|f\|.$$

Hence, it is bounded. (Moreover, it is continuous)

(2) Let's show that it's surjective onto the unit ball of X. Notice that by linearity we will get that it's surjective onto X, and so we are actually done if we can do this.

Pick $x \in B_1(0) \subset X$. We can find $x_{n_1} \in (x_n)$ such that $||x - x_{n_1}|| < 2^{-1}$. Let $y_1 = 2(x - x_{n_1})$. Then we have that $y_1 \in B_1(0)$, since

$$\|y\| = \|2(x - x_{n_1})\| = 2\|x - x_{n_1}\| < 1.$$

Now choose $x_{n_2} \in (x_n)$ such that $\|y_1 - x_{n_2}\| < 2^{-1}$. Furthermore, notice that

$$\|y_1 - x_{n_2}\| = \|2(x - x_{n_1}) - x_{n_2}\| = 2\|x - x_{n_1} - 2^{-1}x_{n_2}\| < 2^{-1},$$

so that

$$\|x - x_{n_1} - 2^{-1}x_{n_2}\| < 2^{-2}.$$

Continue inductively like this. Then we have that

$$\left\| x - \sum_{1}^{n} 2^{1-k} x_{n_k} \right\| < 2^{-n}.$$

Define $f \in \mathcal{L}^1(\mu)$ by

$$f(j) = \begin{cases} 2^{1-k} \text{ if } x_j = x_{n_k} \\ 0 \text{ otherwise.} \end{cases}$$

We see that $f \in \mathcal{L}^1(\mu)$, since

$$\sum f(j) = \sum 2^{1-k} < \infty,$$

and

$$T(f) = \sum f(n)x_n = \sum 2^{1-k} x_{n_k} = x.$$

So T is surjective.

(3) T is continuous, so $\ker(T)$ is closed. T is surjective, so $\mathrm{Im}(T) = X$, which is closed. By the prior problem, $\mathcal{L}^1(\mu)/\ker(T) \cong X$.

□

Problem 186 (Folland 5.37)**.** Let X and Y be Banach spaces. If $T : X \to Y$ is a linear map such that $f \circ T \in X^*$ for every $f \in Y^*$, then T is bounded.

Proof. We would like to establish the closed graph theorem. Let $(x_n, T(x_n)) \to (x, y)$. We want to show that $y = T(x)$. We have that $(f \circ T(x_n)) \to (f \circ T(x))$ for all $f \in Y^*$, and we have that $(f \circ T(x_n)) \to (f \circ y)$. So for all $f \in Y^*$, we have that $f(y) = f(T(x_n))$. We have that Y^* separates points, so this implies that $y = T(x_n)$. □

Problem 187 (Folland 5.38)**.** Let X and Y be Banach spaces, and let (T_n) be a sequence in $L(X, Y)$ such that $\lim T_n x$ exists for every $x \in X$. Let $Tx = \lim T_n x$, then $T \in L(X, Y)$.

Proof. The idea is to use the Steinhaus boundedness principle. We first need to show that T is linear. Let $x, y \in X$, a a scalar, then

$$T(ax + y) = \lim T_n(ax + y) = \lim aT_n(x) + \lim_n T_n(y) = aT(x) + T(y).$$

So it's indeed linear. Now, we need to show that T is bounded. Notice that

$$||T|| \leq \sup_n ||T_n||,$$

and by the Steinhaus boundedness principle we have that

$$||T_n(x)|| < \infty \;\; \forall x \in X \implies \sup_n ||T_n|| < \infty,$$

so

$$||T|| < \infty.$$

Hence, $T \in L(X, Y)$. ☐

Problem 188 (Folland 5.42)**.** Let $E_n \subset C([0,1])$ be the space of functions f such that there is an $x_0 \in [0,1]$ such that $|f(x) - f(x_0)| \leq n|x - x_0|$ for all $x \in [0,1]$.

(1) Prove that E_n is nowhere dense in $C([0,1])$.
(2) Show that the subset of nowhere differentiable functions is residual in $C([0,1])$.

Proof. (1) Recall that a set A is nowhere dense if $(\overline{A})^{\text{o}} = \varnothing$. Per Royden (Exercise 10.20), we should show first that this set is closed. Take a sequence $(f_i) \subset E_n$ where $f_i \to f \in C([0,1])$. We wish to show that $f \in E_n$. From the definition, for each f_i there exists an $x_i \in [0,1]$ such that $|f_i(x) - f_i(x_i)| \leq n|x - x_i|$ for all $x \in [0,1]$. Notice that this is a sequence of points $(x_i) \subset [0,1]$, so we can extract a convergent subsequence $(x_{i_k}) \subset [0,1]$ such that $x_{i_k} \to x_0 \in [0,1]$. We then would like to show that this is the appropriate choice of x_0 for f such that $|f(x) - f(x_0)| \leq n|x - x_0|$ for all $x \in [0,1]$; that is, $f \in E_n$. Since $f_i \to f \in C([0,1])$, we have that for $\epsilon > 0$, there exists N such that for all $n \geq N$, $||f - f_n||_\infty < \epsilon/2$. Furthermore, we can choose $i \geq N$ such that we have $|f_i(x) - f_i(x_0)| \leq n|x - x_0|$ by uniform continuity. Hence, we get that

$$|f(x) - f(x_0)| = |f(x) - f_i(x) + f_i(x) - f_i(x_0) + f_i(x_0) - f(x_0)|$$

$$\leq |f(x) - f_i(x)| + |f_i(x) - f_i(x_0)| + |f_i(x_0) - f(x_0)|.$$

By the infinity norm, we get that this is less than or equal to

$$|f(x) - f(x_0)| \leq 2||f - f_i|| + |f_i(x) - f_i(x_0)| < \epsilon + n|x - x_0|.$$

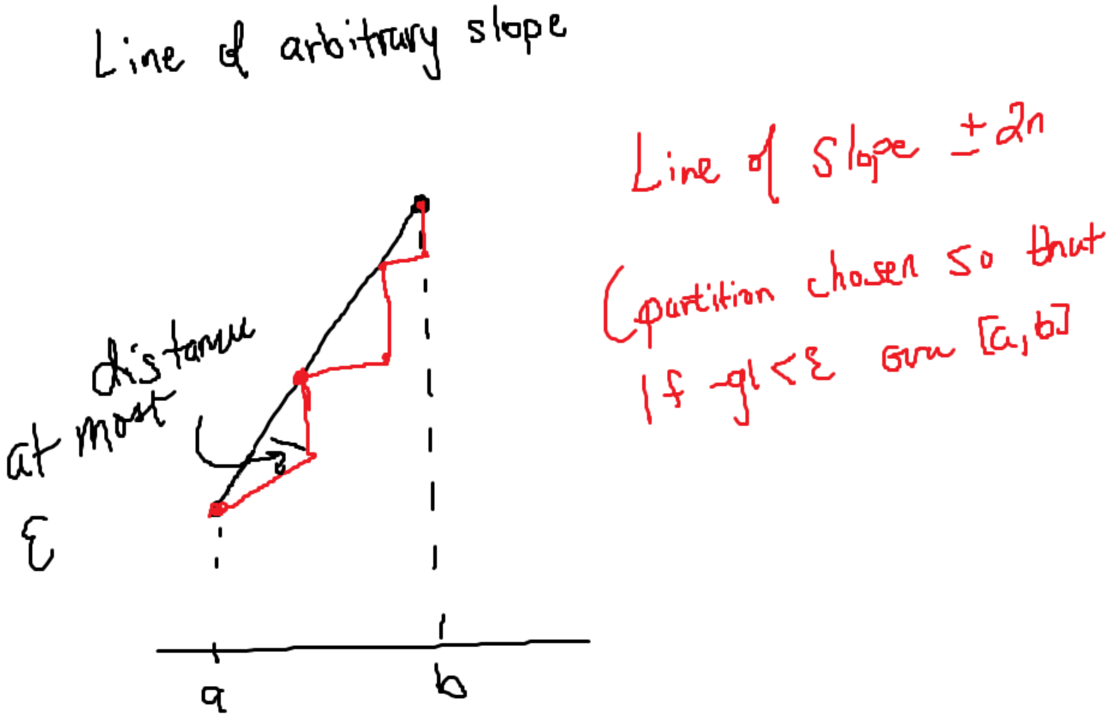
Since this applies for all $\epsilon > 0$, we get that

$$|f(x) - f(x_0)| \leq n|x - x_0|$$

as desired; $f \in E_n$. Hence, E_n is closed, and so it is sufficient to show that $(E_n)^{\text{o}}$ is empty.

By prior homework, we have that PWL are uniformly dense in $C([0,1])$, hence, for all $f \in C([0,1])$, $\epsilon > 0$ fixed, we can find a piecewise linear function h such that $||f - h||_\infty < \epsilon/2$. If we can approximate h with a piecewise linear function g whose linear pieces have slope $\pm 2n$ such that $||h - g||_\infty < \epsilon/2$, then we get that $||f - g||_\infty < \epsilon$, and so we can uniformly approximate $f \in C([0,1])$ with piecewise linear functions whose linear pieces have slope $\pm 2n$. It suffices to show that we can find such a g for a line of arbitrary slope, since we can just subdivide the problem into the piecewise linear components and apply this appropriately.

To see that we can find such a g for a line of arbitrary slope on the interval $[a, b]$, we can do a sort of oscillating approach, drawn below. There will only be finitely many components, and we see that by subdividing it up more, we can approximate it arbitrarily well.



Thus, we can find g with components whose slope is $\pm 2n$ where $||f-g||_\infty < \epsilon$. To show that E_n has empty interior, we need to show that we cannot find an $\epsilon > 0$ such that the ball centered at any $f \in E_n$ with radius ϵ is contained in E_n. Take $f \in E_n$, fixed $\epsilon > 0$, and examine $B_\epsilon(f)$. Then this is the collection of $g \in C([0,1])$ such that $||f-g||_\infty < \epsilon$. With our construction before, we see that we can fined a piecewise linear function g with components whose slopes are $\pm 2n$ such that $g \in B_\epsilon(f)$. But notice that this means that $g \notin E_n$. Hence, we see that $B_\epsilon(f) \not\subset E_n$ for any $\epsilon > 0$, any $f \in E_n$, and so $(E_n)^{\text{o}} = \varnothing$. Hence, it is nowhere dense.

(2) Recall that being residual means being the complement of a meager set. Recall that being meager means that you are a union of nowhere dense sets. Take $g \in C([0,1])$ such that it is differentiable at a point $x \in [0,1]$. This means that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $y \in [0,1]$ with $|y-x| < \delta$, we have

$$\left|\frac{g(y)-g(x)}{y-x} - g'(x)\right| < \epsilon,$$

or

$$|g(y)-g(x)| - |y-x||g'(x)| \le |g(y)-g(x)-(y-x)g'(x)| < |y-x|\epsilon.$$

Choosing $\epsilon = 1$, we have that

$$|g(y)-g(x)| < |y-x|(|g'(x)|+1).$$

So for $|y-x| \ge \delta$, we get that

$$|g(y)-g(x)| \le |g(y)|+|g(x)| \le ||g||_\infty + ||g||_\infty = 2||g||_\infty,$$

so

$$|g(y)-g(x)| \le 2\frac{|y-x|}{\delta}||g||_\infty.$$

Hence, letting $n = \max\{2\delta^{-1}||g||_\infty, |g'(x)|+1\}$, we have that

$$|g(y) - g(x)| \leq n|y - x|$$

for all $y \in [0,1]$. Thus, $g \in E_n$ for some n sufficiently large. To be nowhere differentiable means that g is not differentiable at any point, and so $g \notin E_n$ for any n. Hence, $g \notin \bigcup_{n=1}^\infty E_n$, or $g \in (\bigcup_{n=1}^\infty E_n)^c$. But since this works for all nowhere differentiable functions, we have that $\Gamma \subset (\bigcup_{n=1}^\infty E_n)^c$. In other words, Γ is the complement of a meager set, and so residual. □

Problem 189. Suppose X and Y are Banach.

(1) Show that if $T \in L(X, Y)$ is bounded, then T is weak-weak continuous.
(2) Show that if a linear map $T : X \to Y$ is norm-weak continuous, then $T \in L(X, Y)$.

Proof. (1) Recall that weak convergence says that for all $\varphi \in X^*$, $\varphi(x_n) \to \varphi(x)$. Let $x_n \to x$ weakly. Taking T bounded, we wish to show that, for all $\varphi \in Y^*$, $\varphi(T(x_n)) \to \varphi(T(x))$. Notice that $\varphi \circ T : X \to F$, so $\varphi \circ T \in X^*$. Hence, $\varphi(T(x_n)) \to \varphi(T(x))$.

(2) We wish to show it is bounded; hence, we wish to use the closed graph theorem. Let $x_n \to x$, $T(x_n) \to y$, then we wish to show that $T(x) = y$. Since we have norm-weak continuity, we have that for all $\varphi \in Y^*$, $\varphi(T(x_n)) \to \varphi(T(x))$. Since $\varphi \in Y^*$ is continuous, we have that $\varphi(T(x_n)) \to \varphi(y)$ as well. Since this applies for all $\varphi \in Y^*$, and Y^* separates points, we get that we must have $y = T(x)$, as desired. So T is bounded by the closed graph theorem. □

Problem 190. If X is finite dimensional, prove that the weak topology is the norm topology .

Proof. X is finite dimensional, so we have that $X \cong F^n$, where F the underlying field. Without loss of generality, it suffices to prove the statement for F^n. Let π_i denote the canonical projection maps onto F; these are clearly continuous, and so in $L(F^n, F)$. Consider an open set $B_\epsilon(x)$ under the norm topology. Then we have that

$$B_\epsilon(x) = \{y \in X \ : \ \|x - y\| < \epsilon\}.$$

We can rewrite this as

$$B_\epsilon(x) = \bigcap_{i=1}^n \pi^{-1}(U_\epsilon(x)).$$

Since both of these generate the topologies, we have the open sets with regards to either topology are open with respect to the other topology. Hence, they have equivalent topologies. □

Problem 191. Prove that if X is not finite dimensional, then the weak topology is different from the norm topology.

Proof. Consider the open ball of radius one centered at 0 with respect to the norm; i.e.

$$B_1(0) = \{x \in X \ : \ ||x|| < 1\}.$$

We wish to show that this is not open with respect to the weak topology. Assume for contradiction it were. Then we have that there are linear functionals $f_1, \ldots, f_n$ such that

$$\{x \in X \ : \ |f_k(x)| < 1 \ \text{ for } k = 1, \ldots, n\} \subset B_1(0).$$

That is, we have that for all x such that $|x| < 1$, $|f_k(x)| < 1$. Construct a linear function $f : X \to F^n$ via $f(x) = (f_1(x), \ldots, f_n(x))$. We have then that we can find a vector x with $f(x) = 0$. We can scale this vector to get that $B_1(0)$ is unbounded, which is a contradiction. □

Remark. Chaining the last two statements together, we have that the weak and norm topology are equivalent iff X is finite dimensional.

Problem 192 (Folland 5.47)**.** Suppose that X and Y are Banach spaces.

(1) If $(T_n) \subset L(X,Y)$ and $T_n \to T$ weakly, then $\sup_n ||T_n|| < \infty$.
(2) Every weakly convergent sequence in X, and every weak*-convergent sequence in X^*, is bounded.

Proof. (1) Let $\hat{T}$ denote the adjoint of T; that is, the map so that $f(T) = \hat{T}(f)$. Since $T_n \to T$ weakly, we have that for all $f \in Y^*$, $f(T_n(x)) \to f(T(x))$. We can rewrite this as

$$f(T_n(x)) = \hat{x} \circ \hat{T_n}(f),$$

$$\hat{x} \circ \hat{T_n}(f) \to \hat{x} \circ \hat{T}(f).$$

Fix $x \in X$. Since this converges for all $f \in Y^*$, we get that

$$||\hat{x} \circ \hat{T_n}(f)|| < \infty$$

for all $f \in Y^*$, and so

$$||\hat{x} \circ \hat{T_n}|| < \infty$$

by the Uniform Boundedness principle. Fix $n \in \mathbb{N}$. If $T_n(x) = 0$, we are done. Otherwise, by Hahn-Banach, we have that there exists a linear functional f so that

$$f(T(x)) = ||T(x)||.$$

In other words, we get

$$||T_n(x)|| = ||f(T_n(x))|| = ||\hat{x} \circ \hat{T_n}(f)|| \leq ||\hat{x} \circ \hat{T_n}|| < \infty.$$

Since this applies for all $x \in X$, $n \in \mathbb{N}$, we get that the uniform boundedness principle applies to give us

$$\sup_n ||T_n|| < \infty.$$

(2) Let $x_n \to x$ weakly. Then fixing $f \in X^*$, we have that $f(x_n) \to f(x)$. Hence, we get that

$$||f(x_n)|| = ||\hat{x_n}(f)|| < \infty$$

for all $f \in X^*$. Uniform boundedness applies to give us $||\hat{x_n}|| < \infty$. $\hat{}$ is an isometry, so $||\hat{x_n}|| = ||x_n|| < \infty$.

Let $f_n \to f$ in the weak* topology. Then we have that, for all $x \in X$, $f_n(x) \to f(x)$. In other words, $||f_n(x)|| < \infty$ for all n, and so Uniform Boundedness applies to give us $\sup_n ||f_n|| < \infty$.

□

Problem 193 (Folland 5.48)**.** Suppose that X is a Banach space.

(1) The norm-closed unit ball $B = \{x \in X \ : \ ||x|| \leq 1\}$ is also weakly closed.
(2) If $E \subset X$ is bounded (with respect to the norm), so is its weak closure.
(3) If $F \subset X^*$ is bounded (with respect to the norm), so is its weak* closure.

Proof. (1) Let $x_n \to x$ weakly, $(x_n) \subset B$. We wish to show that $x \in B$. Weak convergence implies that for all $f \in X^*$, $f(x_n) \to f(x)$. We can rewrite this as $\hat{x_n}(f) \to \hat{x}(f)$. Notice that

$$||\hat{x}|| = \sup\{||\hat{x}(f)|| \ : \ f \in X^*\}.$$

Taking f such that $||f|| = 1$, we get that

$$||\hat{x}(f)|| = ||f(x)|| = ||\lim_n f(x_n)|| = ||\lim_n \hat{x_n}(f)|| \leq 1.$$

Since this holds for all $||f|| = 1$, we get that $||\hat{x}|| \leq 1$, and the isometry gives us $||x|| \leq 1$. Hence, it's closed.

(2) Let E be bounded. Then for all $x \in E$, we have that $||x|| \leq M$ for some $M > 0$. Take a specific $x \in E$, then we have that

$$||x|| \leq M \implies \frac{||x||}{M} \leq 1.$$

Hence, $M^{-1}E \subset B$. Taking it's closure, we have that $\overline{M^{-1}E} \subset B$. Multiplication is weakly-continuous, so $M^{-1}\overline{E} \subset \overline{M^{-1}E} \subset B$. Hence, $\overline{E}$ is bounded by M.

(3) Let $f_n \to f$ in the weak* topology. Then for all $x \in X$, $f_n(x) \to f(x)$. Assuming that $||f_n|| \leq M$ for all n, we'd like to show that $||f|| \leq M$. Choose x with $||x|| = 1$, then

$$||f(x)|| = ||\lim_n f_n(x)|| = \lim ||f_n(x)|| \leq \sup_n ||f_n(x)|| \leq \sup_n ||f_n|| \leq M.$$

Since this applies for all x, we get $||f|| \leq M$ as well.

□

Problem 194 (Folland 5.53)**.** Suppose that X is a Banach space and (T_n), (S_n) are sequences in $L(X, X)$ such that $T_n \to T$ strongly and $S_n \to S$ strongly.

(1) If $(x_n) \subset X$ and $||x_n - x|| \to 0$, then $||T_n x_n - Tx|| \to 0$.

(2) $T_n S_n \to TS$ strongly.

Proof. (1) We have

$$\begin{aligned} ||T_n x_n - T_n x + T_n x - Tx|| &\leq ||T_n(x_n - x)|| + ||\hat{x}(T_n - T)|| \\ &\leq ||T_n|| \cdot ||x_n - x|| + ||\hat{x}|| \cdot ||T_n - T||. \end{aligned}$$

Notice that $\sup_n ||T_n||$ is finite by earlier, so we get

$$||T_n|| \cdot ||x_n - x|| + ||\hat{x}|| \cdot ||T_n - T|| \leq M \cdot ||x_n - x|| + ||\hat{x}|| \cdot ||T_n - T||.$$

Letting the sequences go to 0, we win.

(2) Notice that, for fixed $x \in X$, $S_n(x) \to S(x)$, and from the first part we get that $T_n(S_n(x)) \to T_n(S(x)) \to T(S(x))$.

□

Problem 195. Show that $\langle \cdot, \cdot \rangle$ is self-adjoint iff $\langle x, x \rangle \in \mathbb{R}$ for all $x \in H$.

Proof. ($\implies$) If $\langle \cdot, \cdot \rangle$ is self adjoint, then this means that $\langle x, x \rangle = \overline{\langle x, x \rangle}$, which can only happen if $\langle x, x \rangle \in \mathbb{R}$.

($\impliedby$) Suppose $\langle x, x \rangle \in \mathbb{R}$ for all $x \in H$. We wish to show that $\langle x, y \rangle = \overline{\langle y, x \rangle}$. Write $x = a + bi$, $y = c + di$, where $a, b, c, d \in \mathbb{R}$. Then

$$\begin{aligned} \langle x, y \rangle = \langle a + bi, c + di \rangle &= \langle a, c + di \rangle + i \langle b, c + di \rangle \\ &= \langle a, c \rangle - i \langle a, d \rangle + i \langle b, c \rangle + \langle b, d \rangle \\ &= \langle c, a \rangle - i \langle d, a \rangle + i \langle c, b \rangle + \langle d, b \rangle, \\ \overline{\langle y, x \rangle} = \langle c - di, a - bi \rangle &= \langle c, a - bi \rangle - i \langle d, a - bi \rangle \\ &= \langle c, a \rangle + i \langle c, b \rangle - i \langle d, a \rangle + \langle d, b \rangle, \end{aligned}$$

and we see that they are equal. □

Problem 196. Prove that if $\langle \cdot, \cdot \rangle$ is positive (i.e. $\langle x, x \rangle \geq 0$ for all $x \in H$), then it is self-adjoint.

Proof. Follows by the prior problem ($\langle x, x \rangle \in [0, \infty) \subset \mathbb{R}$). □

Problem 197. Suppose H is a Hilbert space, $S, T : H \to H$ are linear operators such that for all $x, y \in H$, $\langle Sx, y \rangle = \langle x, Ty \rangle$.

(1) Prove that $||S|| = ||T||$.

(2) Prove that S and T are bounded.

Proof. (1) We have that

$$||S|| = \sup\{||S(x)|| \ : \ ||x|| = 1\}.$$

Recall that

$$\|S(x)\|^2 = |\langle S(x), S(x)\rangle| = |\langle x, T(S(x))\rangle| \le \|x\|\|T(S(x))\| \le \|T\|\|S\|.$$

So

$$\|S\|^2 \le \|T\|\|S\|,$$

which gives us that

$$\|S\| \le \|T\|.$$

An analogous argument gives

$$\|T\| \le \|S\| \implies \|S\| = \|T\|.$$

(2) Again, we want to use closed graph theorem. Let $x_n \to x$, consider $S(x_n) \to y$. We want to show that $S(x) = y$. We have that, for all $z \in H$,

$$\langle S(x_n), z\rangle = \langle x_n, T(z)\rangle \to \langle x, T(z)\rangle = \langle S(x), z\rangle,$$

and

$$\langle S(x_n), z\rangle \to \langle y, z\rangle.$$

So

$$\langle S(x) - y, z\rangle = 0$$

for all $z \in H$. Recall that $H^\perp \cap H = 0$, so we have that $S(x) = y$. Hence, closed graph theorem tells us that S is bounded. (1) tells us that T is bounded.

□

Problem 198. Suppose $X_0 \subset X$, X Banach, is a dense subspace. Let $T_0 : X_0 \to Y$ be such that T_0 bounded, Y Banach. Then there exists a unique extension $T : X \to Y$ bounded such that $||T|| = ||T_0||$ and $T|_{X_0} = T_0$.

Proof. Let $x \in X$, and take a sequence $(x_n) \subset X_0$ such that $x_n \to x$. Since T is linear and bounded, we get that

$$||T_0(x_n) - T_0(x_m)|| \le C||x_n - x_m|| \to 0,$$

so $(T_0(x_n))$ is a Cauchy sequence, and we have that $T_0(x_n) \to y$. Define $T : X \to Y$ by $T(x) = y$, where y is given as above. We need to check four things.

(1) The definition of T is independent of the sequence: Let (x_n), (y_n) be two sequences such that $x_n \to x$, $y_n \to x$. Then we have

$$\lim_{n\to\infty} (x_n - y_n) = 0,$$

$$\lim_{n\to\infty} T_0(x_n - y_n) = \lim_{n\to\infty} T_0(x_n) - \lim_{n\to\infty} T_0(y_n) = 0,$$

so

$$\lim_{n\to\infty} T_0(x_n) = \lim_{n\to\infty} T_0(y_n) = T(x).$$

(2) T is well-defined: Assume $x_1 = x_2$. Then take a sequence (z_n) such that $z_n \to x_1$, (g_n) such that $g_n \to x_2$. Then we have

$$T(x_1) = \lim_{n\to\infty} T_0(z_n),$$

$$T(x_2) = \lim_{n\to\infty} T_0(g_n),$$

$$T(x_1) - T(x_2) = \lim_{n\to\infty} T_0(z_n - g_n) = T_0(x_1 - x_2) = 0,$$

so $T(x_1) = T(x_2)$.
(3) T is linear, and is an extension of T_0: Let $x, y \in X$, $a \in F$. Then we need to show that $T(ax+y) = aT(x) + T(y)$. Take (x_n) such that $x_n \to x$, (y_n) such that $y_n \to y$. Then we have that $ax_n + y_n \to ax + y$, and so we see

$$T(ax+y) = \lim_{n\to\infty} T_0(ax_n+y_n) = \lim_{n\to\infty} aT_0(x_n)+T_0(y_n) = a \lim_{n\to\infty} T_0(x_n)+ \lim_{n\to\infty} T_0(y_n) = aT(x)+T(y).$$

Let $x \in X_0$, take a sequence (x_n) such that $x_n \to x$, then we have

$$T_0(x) = \lim_{n\to\infty} T_0(x_n) = T(x).$$

(4) T is bounded: Let $x \in X$, then we have a sequence $(x_n) \subset X_0$ so that $x_n \to x$, and we have that

$$||T(x)|| = \lim_{n\to\infty} ||T_0(x_n)|| \leq \lim_{n\to\infty} C||x_n|| = C||x||,$$

by norm continuity.

Next, we need to show that $||T|| = ||T_0||$. Clearly $||T_0|| \leq ||T||$, so it suffices to show that $||T|| \leq ||T_0||$. This follows simply by noting that

$$||T|| = \sup\{||T(x)|| \ : \ ||x|| = 1\} \leq C = \sup\{||T_0(x)|| \ : \ ||x|| = 1, x \in X_0\}.$$

□

Problem 199 (Folland 5.56)**.** If E is a subset of a Hilbert space H, $(E^\perp)^\perp$ is the smallest closed subspace of H containing E.

Proof. In other words, we want that $\overline{E} = (E^\perp)^\perp$.
Step 1: We first show that $E \subset (E^\perp)^\perp$. Recall that

$$E^\perp = \{x \in H \ : \ \langle x, y\rangle = 0 \text{ for all } y \in E\}.$$

If we take $x \in E$, we see that, for all $y \in E^\perp$, $\langle x, y\rangle = 0$, so $x \in (E^\perp)^\perp$. Hence, $E \subset (E^\perp)^\perp$.
Step 2: We now want to show that $(E^\perp)^\perp$ is closed. Let $(y_n) \subset (E^\perp)^\perp$ such that $y_n \to y$. Then we have that, for all $x \in E^\perp$,

$$\langle y, x\rangle = \langle \lim_n y_n, x\rangle = \lim_n \langle y_n, x\rangle = 0.$$

Hence, $y \in (E^\perp)^\perp$, so it is closed. Notice this argument also shows that $E^\perp$ is closed.
Step 3: We want to show that if $S \subset T$, then $T^\perp \subset S^\perp$. This follows from simply noting that if $x \in T^\perp$, then $\langle x, y\rangle = 0$ for all $y \in T$, which in particular means for all $y \in S$, and so $x \in S^\perp$.
Step 4: We now want to show that $\overline{E}^\perp = E^\perp$. Clearly $E^\perp \subset \overline{E}^\perp$. For the other direction, note that

$$E \subset E^{\perp\perp} \implies E^{\perp\perp\perp} \subset E^\perp,$$

and

$$\overline{E}^\perp \subset E^{\perp\perp\perp},$$

so $E^\perp = \overline{E}^\perp$.
Step 4: Write

$$H = \overline{E} \oplus E^\perp.$$

Then for every element $y \in H$, we have that it decomposes uniquely into (x, m), where $x \in \overline{E}$ and $m \in E^\perp$. Hence, for every $y \in E^{\perp\perp}$, we have that $y = x + m$. We wish to show that $m = 0$. Notice that for $z \in E^\perp$, we have

$$0 = \langle y, z\rangle = \langle x + m, z\rangle = \langle x, z\rangle + \langle m, z\rangle = \langle m, z\rangle.$$

Since this applies for all $z \in E^{\perp}$, taking in particular m, we have

$$0 = \langle m, m \rangle,$$

which implies that $m = 0$. Hence, $y \in \overline{E}$, but this implies that $E^{\perp\perp} \subset \overline{E}$, which gives us equality. $\square$

Problem 200 (Folland 5.58)**.** Let M be a closed subspace of the Hilbert space H, and for $x \in H$ let Px be the element of M such that $x - Px \in M^{\perp}$. Show that $P \in L(H, H)$.

Proof. Let $x, y \in H$. We want to show that $P(x+y) = P(x) + P(y)$. Notice that $Px, Py \in \mathcal{M}$, and we have that

$$x + y - (Px - Py) \in M^{\perp},$$

so by uniqueness we must have that $P(x+y) = P(x) + P(y)$. Similarly, $P(ax)$ is an element of $\mathcal{M}$, and we have that

$$ax - a(P(x)) = a(x - P(x)) \in \mathcal{M}^{\perp},$$

so $P(ax) = aP(x)$ by uniqueness. Hence, P is linear. We now want to show that P is bounded.

Take $x \in \mathcal{M}$. Then we have that $x = m + y$, where $m \in \mathcal{M}$, $y \in \mathcal{M}^{\perp}$. Hence, we can identify $P(x) = y$. Notice that

$$||P(x)|| = ||y|| \leq ||x||,$$

so it is bounded, and hence continuous. $\square$

Problem 201. Let H be a Hilbert space. Show that a closed subspace $S \subset H$ is a Hilbert space, with operation inherited from H.

Proof. It suffices to show that it's complete with respect to the norm given by

$$\|x\| = \sqrt{\langle x, x \rangle}.$$

Let $(x_n) \subset S$ be a Cauchy sequence. Then $x_n \to x$ by the completeness of H, but since S closed this implies that $x \in S$. $\square$

Problem 202. Suppose I is a positive linear functional on $C_c(X)$, for each compact $K \subset X$ there is a constant C_K such that

$$|I(f)| \leq C_K ||f||_u$$

for all $f \in C_c(X)$ such that $\text{supp}(f) \subset K$.

Proof. Take $f \in C_c(X)$. It suffices to look at real valued f by breaking it up into the real and complex parts. Now, use Urysohn to find $\phi \in C_c(X, [0,1])$ such that $\phi|_K = 1$. Then we have that, if $\text{supp}(f) \subset K$,

$$|f| \leq ||f||_u \phi.$$

Then

$$-f \leq ||f||_u \phi, \quad f \leq ||f||_u \phi,$$

so

$$0 \leq ||f||_u \phi + f, \quad 0 \leq ||f||_u \phi - f,$$

so since I is a positive linear functional we have

$$I(||f||_u \phi + f) = I(||f||_u \phi) + I(f) \geq 0,$$
$$I(||f||_u \phi - f) = I(||f||_u \phi) - I(f) \geq 0,$$

so

$$|I(f)| \leq ||f||_u I(\phi).$$

$\square$

Problem 203 (Folland 7.1)**.** Let X be a LCH space, Y a closed subset of X, and μ a Radon measure on Y. Then $I(f) = \int (f|_Y) d\mu$ is a positive linear functional on $C_c(X)$, and the induced Radon measure ν on X is given by $\nu(E) = \mu(E \cap Y)$.

Proof. Linearity follows from the fact that

$$\int (f|_Y) d\mu = \int_Y f d\mu$$

and the linearity of the integral. For positivity, take $f \geq 0$. Measures are positive, so we see that

$$\int_Y f d\mu \geq 0.$$

Hence, it's a positive linear functional on $C_c(X)$. Notice that we can write

$$I(f) = \int f \cdot \chi_Y d\mu.$$

Notice there exists a positive Radon measure ν on X such that

$$I(f) = \int f d\nu.$$

Take $E \in \mathcal{M}$, $f = \chi_E$. Then we have

$$I(\chi_E) = \int \chi_E \cdot \chi_Y d\mu = \mu(E \cap Y),$$

$$I(\chi_E) = \int \chi_E d\nu = \nu(E),$$

so we have

$$\nu(E) = \mu(E \cap Y).$$

□

Problem 204 (Folland 7.2)**.** Let μ be a Radon measure on X.

(1) Let N be the union of all open $U \subset X$ such that $\mu(U) = 0$. Then N is open and $\mu(N) = 0$. The complement of N is called the support of μ.

(2) $x \in \operatorname{supp}(\mu)$ iff $\int f d\mu > 0$ for every $f \in C_c(X, [0,1])$ such that $f(x) > 0$.

Proof. (1) Clearly N is open (arbitrary union of open sets is open). Using the fact that μ is a Radon measure, since N is open, we have that

$$\mu(N) = \sup\{\mu(K) \ : \ K \subset N, K \text{ compact.}\}.$$

Let $K \subset N$, K compact. Then we have

$$K \subset \bigcup U_\alpha,$$

where U_α is such that $\mu(U_\alpha) = 0$. Since K is compact, we have

$$K \subset \bigcup_{j=1}^{n} U_j.$$

Hence,

$$\mu(K) = \mu\left(\bigcup_{j=1}^{n} U_j\right) \leq \sum_{j=1}^{n} \mu(U_j) = 0.$$

Since this applies for all K, we get $\mu(N) = 0$.

(2) Take $x \in \operatorname{supp}(\mu) = N^c$. Let $f(x) = c > 0$. Since f is continuous, we have $f^{-1}((c/2, 1))$ is an open set (say V), and we have $x \in V$. Since V is an open set, $V \not\subset N$, we must have $\mu(V) > 0$. Hence,

$$\int f d\mu \geq \int_V f d\mu > \frac{c}{2}\mu(V) > 0.$$

Now, assume that

$$\int f d\mu > 0$$

for every $f \in C_c(X, [0,1])$ such that $f(x) > 0$. Notice $\{x\}$ is a compact set, take an open set U such that $x \in U$. Use LCH Urysohn to get a $g \in C_c(X, [0,1])$ so that $g(x) = 1$ and $\operatorname{supp}(g) \subset U$. Then

$$\mu(U) \geq \int g d\mu > 0.$$

□

Problem 205 (Folland 7.3)**.** Let X be the one-point compactification of a set with the discrete topology. If μ is a Radon measure on X, then $\operatorname{supp}(\mu)$ is countable, where $\operatorname{supp}(\mu) = N^c$, with N being the union of all open U such that $\mu(U) = 0$.

Proof. Let Y be a set, $\tau = \mathcal{P}(Y)$ the discrete topology on Y. Let X be the one-point compactification of Y; i.e. the open sets are subsets of X or sets with infinity such that their complements is open. We have that μ is a Radon measure, which means that it's finite on compact sets, outer regular on all Borel sets, inner regular on open sets. Examine

$$N := \operatorname{supp}(\mu)^c = \bigcup \{U \ : \ U \subset \mathcal{P}(Y),\ \mu(U) = 0\}.$$

For all $y \in Y$ such that $\mu(\{y\}) = 0$, we have that $y \in N$. So

$$Y \cap N^c = \{y \in Y \ : \ \mu(y) > 0\}.$$

Since X is compact, μ finite on all compact sets, $\mu(X) < \infty$. If $\operatorname{supp}(\mu)$ was uncountable, then $\operatorname{supp}(\mu) \cap Y$ is also uncountable. So

$$\mu(Y) \geq \mu(\operatorname{supp}(\mu) \cap Y) = \sum_{y \in \operatorname{supp}(\mu) \cap Y} \mu(\{y\}) = \infty.$$

This is a contradiction. □

Problem 206 (Folland 7.4)**.** Let X be a LCH space.

(1) If $f \in C_c(X, [0, \infty))$, then $f^{-1}([a, \infty))$ is a compact G_δ set for all $a > 0$.
(2) If $K \subset X$ is a compact G_δ set, there exists $f \in C_c(X, [0,1])$ such that $K = f^{-1}(\{1\})$.

Proof. (1) We have

$$f^{-1}([a, \infty)) = \bigcap_{n=1}^{\infty} f^{-1}((a - 1/n, \infty)),$$

where $f^{-1}((a - 1/n, \infty))$ are open sets. So this is a G_δ set. To see it's compact, we notice that $f^{-1}([a, \infty))$ is a closed set (by continuity, using the subspace topology), so we have a closed subset of a compact set, which is then compact.

(2) Since K is a G_δ set, we can write

$$K = \bigcap_{i=1}^{\infty} U_i.$$

For each i, we have that we can find precompact open W so that

$$K \subset W \subset \overline{W} \subset U_1.$$

Write

$$V_n = \left(\bigcap_{i=1}^{n} U_i \right) \cap W.$$

This is an open subset for each n. Using LCH Urysohn's lemma, for each n, we can find $f_n \in C_c(X)$ so that $0 \leq f_n \leq 1$, $\text{supp}(f_n) \subset V_n$, $f = 1$ on K. Write

$$f = \sum_{1}^{\infty} \frac{1}{2^k} f_k.$$

We have $\text{supp}(f) \subset \overline{W}$, so $f \in C_c(X, [0,1])$. If $x \notin K$, then we have that there is some n so that $x \notin V_n$, so $f_n(x) = 0$, and therefore $f(x) < 1$. So $K = f^{-1}(\{1\})$.

□

Problem 207. If μ is a σ-finite Radon meausre on X and $E \subset X$ is Borel, then for all $\epsilon > 0$, there exists a $F \subset E \subset U$ with F closed and U open such that $\mu(U - F) < \epsilon$.

Proof. Suppose $\mu(E) < \infty$, μ is outer regular on E, fix $\epsilon > 0$. We can find U such that $\mu(U - E) < \epsilon/2$. We can then by **Proposition 7.5** find closed F so that $\mu(E - F) < \epsilon/2$. Then $\mu(U - F) < \mu(U - E) + \mu(E - F) < \epsilon$.

Suppose $\mu(E) = \infty$. We have $E = \bigcup E_j$, $\mu(E_j) < \infty$. Find U_j such that $E_j \subset U_j$, $\mu(U_j - E_j) < \epsilon 2^{-j-1}$. Then

$$\mu(U - E) < \sum \mu(U_j - E_j) < \sum \epsilon 2^{-j-1} < \epsilon/2.$$

Apply the same argument to E^c to find V with $\mu(V - E^c) < \epsilon/2$, and so we get $\mu(E - V^c) < \epsilon/2$, and $V^c \subset E$. □

Problem 208 (Folland 7.7)**.** If μ is a σ-finite Radon measure on X and $A \in \mathcal{B}_X$, the Borel measure μ_A defined by $\mu_A(E) = \mu(E \cap A)$ is a Radon measure.

Proof. From prior exercises, we know this is a measure. It suffices to show it is a Radon measure; that is, finite on compact sets, outer regular on Borel sets, and inner regular on open sets. Let K be compact. Then $\mu_A(K) = \mu(A \cap K) \leq \mu(K) < \infty$, so it is finite on all compact sets. Let $F \in \mathcal{B}_X$, then we get that μ is outer regular on $A \cap F$. So there is an open set U such that $A \cap F \subset U$, and $\mu(U) < \mu(A \cap F) + \epsilon$.

Suppose $\mu_A(F) = \infty$, then we're done. Otherwise, suppose $\mu_A(F) < \infty$. Write $F \subset (F \cap A) \cup A^c$. We can find open U such that $F \cap A \subset U$, and $\mu(U) < \mu(F \cap A) + \epsilon/2$. So we need to find an open V such that $A^c \subset V$. Use the outer regularity of μ to get $\mu(V - F) < \epsilon/2$, where V is open and F closed, $F \subset A^c \subset V$. We have $U \cup V$ is open, and hence we get

$$\mu_A(U \cup V) = \mu((U \cup V) \cap A) = \mu((U \cap A) \cup (V \cap A)) \leq \mu(U \cap A) + \mu(V \cap A) \leq \mu(U) + \mu(V \cap F^c) < \mu_A(F) + \epsilon.$$

Hence, we get outer regularity.

Finally, we need to get inner regularity. This follows by using the inner regularity of μ. □

Problem 209 (Folland 7.8)**.** Suppose that μ is a Radon measure on X. If $\phi \in L^1(\mu)$ and $\phi \geq 0$, then $\nu(E) = \int_E \phi d\mu$ is a Radon measure.

Proof. Again, we must show the properties that it is finite on compact sets, outer regular on Borel sets, and inner regular on open sets. Let K compact. Then we have that

$$\int_K \phi \leq \int \phi < \infty.$$

Next, we wish to show that it is outer regular on Borel sets. We can use the absolute continuity to get that there is a $\delta > 0$ such that if $\mu(F) < \delta$, then

$$\int_F \phi d\mu = \nu(F) < \epsilon/2.$$

Let E be a Borel set. Let $E_n = \{x : \phi(x) > 1/n\}$. We have that $\mu(E_n) < \infty$. Let $F = \bigcup E_n = \{x : \phi(x) \neq 0\}$, then $E - F = \{x : \phi(x) = 0\}$. Hence, $\nu(E) = \nu(E - F) + \nu(F) = \nu(F)$. Since ν finite, we get that there exists an n so that $\nu(F - E_n) < \epsilon/2$. Find compact $K \subset E_n$ with $\mu(E_n - K) < \delta$. Hence, $\nu(E_n - K) < \epsilon/2$. So $K \subset E_n \subset F \subset E$,

$$\nu(E - K) = \nu(E - F) + \nu(F - E_n) + \nu(E_n - K) < \epsilon,$$

so K is compact and we conclude inner regular.

We now want to show outer regular. Given a Borel set E, we have that there is a compact $K \subset E^c$ with $\mu(E^c - K) < \epsilon$. Hence, $\mu(K^c - E) < \epsilon$, and K^c open. So we get that it's Radon. □

Problem 210. Complete the proof of **Proposition 3.1**.

Proof. Let ν be a signed measure on $(X, \mathcal{M})$. If (E_j) is an increasing sequence in $\mathcal{M}$, then

$$\nu\left(\bigcup E_j\right) = \lim \nu(E_j).$$

If (E_j) is a decreasing sequence, with E_1 finite, then

$$\nu\left(\bigcap E_j\right) = \lim \nu(E_j).$$

□

Proof. Assume first of all that $\bigcup E_j$ is finite; otherwise, we must have that $\nu(E_n)$ is infinite for some n, and so the result is clearly true. Write

$$G_1 = E_1,$$
$$G_2 = E_2 - E_1,$$
$$\dots$$
$$G_n = E_n - \left(\bigcup_1^{n-1} E_j\right).$$

Then we have that

$$\bigcup E_j = \bigsqcup G_j,$$

and furthermore

$$\nu\left(\bigcup E_j\right) = \nu\left(\bigsqcup G_j\right) = \sum \nu(G_j).$$

Notice now that

$$\nu(G_j) = E_j - E_{j-1},$$

so rewriting the right hand side we have

$$\sum \nu(G_j) = \lim \sum_1^n \nu(G_j) = \lim \left(\nu(E_1) + \sum_2^n \nu(E_j) - \nu(E_{j-1})\right).$$

Hence, we have

$$\sum \nu(G_j) = \nu(E_n),$$

and so

$$\nu\left(\bigcup E_j\right) = \lim \nu(E_n).$$

We can use the first part and the finiteness of E_1. Write

$$F_1 = E_1,$$
$$F_2 = E_1 - E_2,$$
$$\dots$$
$$F_n = E_1 - E_n.$$

Then F_n is now an increasing sequence, and so applying the first part we get

$$\nu(E_1) - \nu\left(\bigcap E_n\right) = \nu\left(E_1 - \bigcap E_n\right) = \nu\left(\bigcup F_j\right) = \lim \nu(F_n) = \nu(E_1) - \lim \nu(E_n).$$

The finiteness of E_1 let's us add and subtract things to get the desired result. □

Problem 211 (Folland 3.2)**.**

(1) If ν is a signed measure, E is ν null if and only if $|\nu|(E) = 0$.
(2) If ν and μ are signed measures, TFAE:
   (a) $\nu \perp \mu$,
   (b) $|\nu| \perp \mu$,
   (c) $\nu^+ \perp \mu$, $\nu^- \perp \mu$.

Proof. (1) ($\Longrightarrow$) Assume E is ν null. Write $\nu = \nu^+ - \nu^-$, $X = P \sqcup N$, $\nu^+(N) = \nu^-(P) = 0$. Then $|\nu| = \nu^+ + \nu^-$, and we have

$$|\nu|(E) = \nu^+(E) + \nu^-(E) = \nu(E \cap P) - \nu^-(E \cap N),$$

and since E is ν-null, each respective component is 0. Hence, $|\nu|(E) = 0$.
($\Longleftarrow$) Assume $|\nu|(E) = 0$. Since $|\nu|$ a positive measure, for all $F \subset E$, $|\nu|(F) = 0$. Furthermore, this gives that $\nu^+(F), \nu^-(F) = 0$ for all $F \subset E$. Using this, we get

$$\nu(F) = \nu^+(F) - \nu^-(F) = 0.$$

Hence, E is ν-null.

(2) See HW14.

□

Problem 212 (Folland 3.5)**.** If ν_1, ν_2 are signed measures that both omit the value ∞ or $-\infty$, then $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$.

Proof. Since ν_1, ν_2 are signed measures, we have

$$\nu_1 = \nu_1^+ - \nu_1^-,$$
$$\nu_2 = \nu_2^+ - \nu_2^-.$$

We see that $\nu_1 + \nu_2$ is also a signed measure, and so we write

$$\nu_1 + \nu_2 = \lambda - \mu.$$

Hence,

$$|\nu_1 + \nu_2| = \lambda + \mu.$$

Notice as well that

$$\nu_1 + \nu_2 = \nu_1^+ - \nu_1^- + \nu_2^+ - \nu_2^-.$$

Hence, we have that

$$\nu_1^+ + \nu_1^- \geq \lambda,$$
$$\nu_2^+ + \nu_2^- \geq \mu,$$

so

$$|\nu_1 + \nu_2| = \lambda + \mu \leq \nu_1^+ + \nu_1^- + \nu_2^+ + \nu_2^- = |\nu_1| + |\nu_2|.$$

□

Problem 213. Suppose that ν is a signed measure on $(X, \mathcal{M})$ and $E \in \mathcal{M}$.

(1)
$$\nu_+(E) = \sup\{\nu(F) \ : \ F \in M, F \subset E\}$$
and
$$\nu_-(E) = -\inf\{\nu(F) \ : \ F \in \mathcal{M}, F \subset E\}.$$

(2)
$$|\nu|(E) = \sup\left\{\sum_1^n |\nu(E_j)| \ : \ n \in \mathbb{N}, E_1, \ldots, E_n \text{ are disjoint, and } \bigcup_1^n E_j = E\right\}.$$

Proof. (1) Write
$$\nu = \nu_+ - \nu_-$$
where
$$X = P \sqcup N, \quad \nu_+(N) = \nu_-(P) = 0.$$
We have that, for $E \in \mathcal{M}$,
$$\nu_+(E) = \nu(E \cap P).$$
Notice that P is positive, so we can use this to deduce that
$$\nu_+(F) \leq \nu_+(E)$$
for all $F \in \mathcal{M}$, $F \subset E$. Hence,
$$\sup\{\nu(F) \ : \ F \in \mathcal{M}, F \subset E\} \leq \nu_+(E).$$
For the other direction, use the fact that
$$E \cap P \in \{F \ : \ F \in \mathcal{M}, F \subset E\}$$
to get that
$$\nu_+(E) \leq \sup\{\nu(F) \ : \ F \in M, F \subset E\}.$$
Hence,
$$\nu_+(E) = \sup\{\nu(F) \ : \ F \in M, F \subset E\}.$$
The argument is analogous for $\nu_-(E)$.

(2) Let $E \in \mathcal{M}$. Then we have that
$$|\nu|(E) = |\nu|(E \cap P) + |\nu|(E \cap N) = \nu_+(E \cap P) + \nu_-(E \cap P) + \nu_+(E \cap N) + \nu_-(E \cap P)$$
$$= \nu_+(E \cap P) + \nu_-(E \cap N) = |-\nu_-(E \cap N)| + |\nu_+(E \cap P)| = |\nu(E \cap P)| + |\nu(E \cap N)|.$$
Hence,
$$|\nu|(E) \leq \sup\left\{\sum_1^n |\nu(E_j)| \ : \ n \in \mathbb{N}, E_1, \ldots, E_n \text{ are disjoint, and } \bigcup_1^n E_j = E\right\}.$$
Now, notice that
$$\sum |\nu(E_j)| \leq \sum |\nu_+(E_j)| + |\nu_-(E_j)| = \sum |\nu|(E_j) = |\nu|(E),$$
since $|\nu|$ is a positive measure. Hence,
$$|\nu|(E) \geq \sup\left\{\sum_1^n |\nu(E_j)| \ : \ n \in \mathbb{N}, E_1, \ldots, E_n \text{ are disjoint, and } \bigcup_1^n E_j = E\right\}.$$
$\square$

Problem 214. Let μ be a σ-finite signed measure, show that

$$\left|\frac{d\mu}{d|\mu|}\right| = 1 \ |\mu| \text{ a.e.}$$

Proof. We define

$$d\mu = \frac{d\mu}{d|\mu|}d|\mu|.$$

We first want to show that $\mu \ll |\mu|$. Since μ is a signed measure, we have a decomposition

$$\mu = \mu_+ - \mu_-, \quad X = P \sqcup N, \quad \mu_+(N) = \mu_-(P) = 0.$$

We have then that

$$|\mu| = \mu_+ + \mu_-.$$

Take $E \in \mathcal{M}$ such that

$$|\mu|(E) = 0 = \mu_+(E) + \mu_-(E) = 0.$$

Since μ_+, μ_- are positive measures, we see that this forces $\mu_+(E) = \mu_-(E) = 0$, which then forces $\mu(E) = 0$. So $\mu \ll |\mu|$.

By the LRN theorem, we get that there is a function $\frac{d\mu}{d|\mu|}$ such that

$$d\mu = \frac{d\mu}{d|\mu|}d|\mu|,$$

and it is unique up to $|\mu|$-a.e. Let $f = \chi_P - \chi_N$. Then we have

$$\mu(E) = \mu(E \cap P) + \mu(E \cap N),$$
$$\mu(E \cap P) = \mu_+(E) = \int_E \chi_P d\mu_+ = \int_E \chi_P d|\mu|,$$
$$\mu(E \cap N) = -\mu_-(E) = -\int_E \chi_N d\mu_- = -\int_E \chi_N d|\mu|,$$

so

$$\mu(E) = \int_E (\chi_P - \chi_N) d|\mu|.$$

Hence,

$$\frac{d\mu}{d|\mu|} = f$$

$|\mu|$-a.e., and we see that

$$\left|\frac{d\mu}{d|\mu|}\right| = |f| = 1$$

since $X = P \sqcup N$. □

Problem 215 (Folland 3.10)**.** Theorem 3.5 may fail when ν is not finite.

Proof. Recall the statement of Theorem 3.5:
Let ν be a *finite* signed measure and μ a positive measure on $(X, \mathcal{M})$. Then $\nu \ll \mu$ if and only if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $|\nu(E)| < \epsilon$ whenever $\mu(E) < \delta$.

Consider $d\nu(x) = \frac{dx}{x}$ and $d\mu(x) = dx$ on $(0, 1)$. We first check that $\nu \ll \mu$. Take $E \in \mathcal{M}$ such that

$$\int_E dx = 0 = \mu(E).$$

Then we have that

$$\nu(E) = \int_E \frac{dx}{x} = 0.$$

However, we see that for all δ,

$$\nu((0,\delta)) = \int_0^\delta \frac{dx}{x} = \lim_{m\to 0} \log(\delta) - \log(m) \geq \lim_{m\to 0} -\log(m) = \infty.$$

So ν is not finite. Furthermore, given any $\epsilon > 0$, δ, let $E = (0, \delta/2)$. Then

$$\mu(E) < \delta,$$

while

$$|\nu(E)| = \left| \int_0^{\delta/2} \frac{dx}{x} \right| \to \infty.$$

□

Problem 216 (Folland 3.11)**.** Let μ be a positive measure. A collection of function $\{f_\alpha\} \subset \mathcal{L}^1(\mu)$ is called *uniformly integrable* if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\left| \int_E f_\alpha d\mu \right| < \epsilon$$

for all $\alpha \in A$ whenever $\mu(E) < \delta$.

(1) Any finite subset of $\mathcal{L}^1(\mu)$ is uniformly integrable.
(2) If $\{f_n\}$ is a sequence in $\mathcal{L}^1(\mu)$ that converges in the $\mathcal{L}^1$ metric to $f \in \mathcal{L}^1(\mu)$, then $\{f_n\}$ is uniformly integrable.

Proof. (1) Take $f \in \mathcal{L}^1(\mu)$. Then we have that f is uniformly integrable; to see this, notice that the measure $\nu(E) := \int_E f d\mu$ is a finite signed measure. Furthermore, we see that $\nu \ll \mu$; take $E \in \mathcal{M}$ such that $\mu(E) = 0$, then we have

$$\nu(E) = \int_E f d\mu = \int (f \cdot \chi_E) d\mu = \int 0 d\mu = 0.$$

So by Theorem 3.5, we have the desired result. Now, take a finite collection $\{f_k\}_{k=1}^n$. Then for $\epsilon > 0$, we have $\delta_1, \ldots, \delta_n$ such that

$$\left| \int_E f_k d\mu \right| < \epsilon$$

assuming $\mu(E) < \delta_k$. Taking $\delta = \min\{\delta_1, \ldots, \delta_n\}$, we have the desired result.

(2) We have that

$$||f_n - f||_1 = \int |f_n - f| \to 0.$$

Choose $\epsilon > 0$ and N sufficiently large so that for all $n \geq N+1$,

$$\int |f_n - f| d\mu < \frac{\epsilon}{2}.$$

Let $I = \{j\}_{j=1}^N$. Then we have that $\{f_k\}_{k\in I}$ is uniformly integrable by (a), so we have a δ so that

$$\left| \int_E f_k d\mu \right| < \frac{\epsilon}{2}$$

for $k \in I$. For $k \in \mathbb{N} - I$, we have

$$\left| \int_E f_k d\mu \right| = \left| \int_E (f_k - f) + f d\mu \right| \leq \int_E |f_k - f| d\mu + \left| \int_E f d\mu \right|,$$

and we see that this gives us that by the same δ above, we have

$$\left| \int_E f_k d\mu \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

Problem 217 (Folland 3.12)**.** For $j = 1, 2$ let μ_j, ν_j be σ-finite measures on $(X_j, \mathcal{M}_j)$ such that $\nu_j \ll \mu_j$. Then $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$, and

$$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x_1, x_2) = \frac{d\nu_1}{d\mu_1}(x_1)\frac{d\nu_2}{d\mu_2}(x_2).$$

Proof. For the first one, we let $E \in \mathcal{M}_1 \times \mathcal{M}_2$ be such that $(\mu_1 \times \mu_2)(E) = 0$. Then we have that

$$(\mu_1 \times \mu_2)(E) = \inf\left\{\sum \mu_1(A_i)\mu_2(B_i) \ : \ E \subset \bigcup A_i \times B_i, \ A_i \in \mathcal{M}_1, \ B_i \in \mathcal{M}_2\right\}.$$

Notice for this to be 0, we must have that $\mu_1(A_i)$, $\mu_2(B_i) = 0$ for all i, and so by the absolute continuity of ν_i we get that $\nu_1(A_i), \nu_2(B_i) = 0$, and hence

$$(\nu_1 \times \nu_2)(E) = 0.$$

So $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$.

For the second part, we write

$$(\nu_1 \times \nu_2)(E) = \int_E \frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)} d(\mu_1 \times \mu_2) = \int_E d(\nu_1 \times \nu_2).$$

We rewrite this as

$$\int_E d(\nu_1 \times \nu_2) = \int \chi_E d(\nu_1 \times \nu_2) = \iint \chi_{E^y}(x) d\nu_1(x) d\nu_2(y) = \iint \chi_E(x, y) d\nu_1(x) d\nu_2(y).$$

Using Fubini in multiple successions, we get

$$\iint \chi_E(x, y) d\nu_1(x) d\nu_2(y) = \iint \chi_E(x, y)\frac{d\nu_1}{d\mu_1}(x) d\mu_1(x) d\nu_2(y) = \iint \chi_E(x, y)\frac{d\nu_2}{d\mu_2}(y) d\mu_2(y)\frac{d\nu_1}{d\mu_1}(x) d\mu_1(x)$$

$$= \int_E \frac{d\nu_1}{d\mu_1}(x)\frac{d\nu_2}{d\mu_2}(y) d(\mu_1 \times \mu_2).$$

By uniqueness, we have the result. □

Problem 218 (Folland 3.13)**.** Let $X = [0, 1]$, $\mathcal{M} = \mathcal{B}_{[0,1]}$, m Lebesgue measure, and μ counting measure on $\mathcal{M}$.

(1) $m \ll \mu$ but $dm \neq f d\mu$ for any f.
(2) μ has no Lebesgue decomposition with respect to m.

Proof. (1) We have $\mu(E) = 0$ iff $E = \varnothing$, and $m(\varnothing) = 0$. So $m \ll \mu$. Assume $dm = f d\mu$. Then we have

$$m(E) = \int_E f d\mu.$$

Take $E = \{x\}$, then

$$m(\{x\}) = 0 = \int_{\{x\}} f d\mu = f(x),$$

and since this applies for all x we have that $f = 0$. However, we see

$$\int_X f d\mu = m(X) = 1,$$

which is a contradiction.

(2) Assume that

$$\mu = \lambda + \rho,$$

where $\lambda \perp m$, $\rho \ll m$. Since $\rho \ll m$, we have that $\rho(\{x\}) = 0$. Hence, since $\mu(\{x\}) = 1$, we get that $\lambda(\{x\}) = 1$. This then applies for all $x \in X$. Since $\lambda \perp m$, we have that we can write $X = A \sqcup B$, where A is λ null and B is m null. By the prior remark, we see that it forces $A = \varnothing$, but this is a contradiction since $m(B) = 1$.

$\square$