NOTES FOR MANIFOLDS

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These are notes from when I subbed on 3/23 and 3/25. All mistakes and typos are my own. Unless otherwise stated, the exercises are meant to be educational and will not be graded. If you are interested in feedback, feel free to email me your solutions.

1. Lecture 0: The long line

There were some questions on the long line which I brushed off – if you are interested, here is a more detailed write up. This is not really relevant for the course.

Definition 1. An *order relation* on X is a binary relation $<$ which satisfies the following for all $a, b, c \in X$:

(1) if $a < b$ and $b < c$, then $a < c$,

(2) if $a < b$, then $a \neq b$,

(3) at least one of the following must hold if $a \neq b$:

  (i) $a < b$,

  (ii) $b < a$.

We write $a \leq b$ to indicate $a < b$ or $a = b$. We say that X is *well-ordered* if every non-empty subset Y of X has a smallest element, i.e., an element $a \in Y$ such that for all $b \in Y$, we have $a \leq b$.

Example 1. The real numbers are not well-ordered, since open intervals do not admit a smallest element. The natural numbers, on the other hand, are well-ordered.

We note that such sets come with a natural topology.

Definition 2. Let X be admit an order relation. The *order topology* is the topology generated by sets of the form

$$(a, b) := \{x \in X \mid a < x < b\}, \quad (a, \infty) := \{x \in X \mid a < x\}, \quad (-\infty, a) := \{x \in X \mid x < a\}.$$

The following is a non-trivial consequence of the axiom of choice.

Theorem 1 (Zermelo's well-ordering theorem)**.** *If A is a set, then there is an order relation on A which makes A well-ordered.*

As a corollary, we have that there is an uncountable set X with a well-ordering. We will need two definitions.

Definition 3. Given two well-ordered sets X and Y, the product $X \times Y$ admits the *dictionary ordering*:

$$(x, y) < (x', y') \iff x < y \text{ or } x = x' \text{ and } y < y'.$$

Note that this makes $X \times Y$ a well-ordered set.

Definition 4. Let X be a well-ordered set. Given $a \in X$, we define the *a-section* of X to be

$$X_a := \{x \in X \mid x < a\}.$$

Lemma 2. *There exists a well-ordered set A having a largest element Ω such that the section X_Ω is uncountable, but every other section of A is countable.*

Proof. Let B be an uncountable well-ordered set and consider the subset

$$X := \{x \in B \mid B_x \text{ is uncountable}\}.$$

There are now two cases:

(1) If X is the empty set, let Ω be some element not in B. The ordering extends to the set $B \cup \{\Omega\}$ by saying $b < \Omega$ for all $b \in B$. We then set $A := B \cup \{\Omega\}$.

(2) If X is non-empty, then let $\Omega \in X$ be the smallest element; such an element exists by the well-ordering principle. We then set $A := B_\Omega \cup \{\Omega\}$. □

Let A be the uncountable set coming from the previous lemma and denote its minimal element by 0.

Definition 5. The *long line* is the set

$$L := (A \times [0,1)) \smallsetminus \{(0,0)\}.$$

We equip this with the *dictionary order topology*, i.e., the order topology coming from the dictionary ordering.

Exercise 1. Show that every ordered set equipped with the order topology is Hausdorff.

Lemma 3. *For every $(\alpha, t) \in L$, there is an open set U containing (α, t) which is homeomorphic to $\mathbb{R}$.*

Proof. There are two cases.

(1) If $t > 0$, then consider the open interval

$$((\alpha,0),(\alpha,1)) = U := \{(\alpha,t) \mid 0 < t < 1\}.$$

There is an obvious homeomorphism to $\mathbb{R}$.

(2) If $t = 0$, then $\alpha > 0$. Note that there exists $\alpha' < \alpha < \alpha''$, and hence

$$(\alpha,0) \in U := ((\alpha',1/2),(\alpha'',1/2)).$$

Because sections of A are countable, one can view U as countably gluing open intervals together, which is homeomorphic to $\mathbb{R}$.

□

Exercise 2. Show that L is *not* second countable.
Hint: Show that any countable subset of L is bounded above by some element in L. Use this to show that there cannot be a countable dense subset.

2. Lecture 1

The goal of the study of (smooth) manifolds is to describe spaces where Euclidean calculus makes sense.

Example 2. Consider the equivalence class on $\mathbb{R}^3 \setminus \{0\}$ given by

$$v \sim w \iff v = \lambda w \text{ for some } \lambda \neq 0.$$

Since any vector can be identified with its endpoint, we are able to introduce *coordinates* on the space $\mathbb{RP}^2 := (\mathbb{R}^3 \setminus \{0\})/\sim$ in the following fashion:

$$[x : y : z] = \{\lambda \cdot (x, y, z) \mid \lambda > 0\}.$$

Note that if $z > 0$, then

$$[x : y : z] = \left[\frac{x}{z} : \frac{y}{z} : 1\right].$$

Thus, there are three *coordinate neighborhoods* of interest on $\mathbb{RP}^2$:

$$\begin{cases} U_1 := \{[x : y : 1] \mid x, y \in \mathbb{R}\}, \\ U_2 := \{[x : 1 : z] \mid x, z \in \mathbb{R}\}, \\ U_3 := \{[1 : y : z] \mid y, z \in \mathbb{R}\}, \end{cases}$$

In particular,

$$\mathbb{RP}^2 = \bigcup_{i=1}^{3} U_i.$$

Also note that each neighborhood is homeomorphic to $\mathbb{R}^2$ in obvious fashions. For example,

$$\varphi_1 : U_1 \to \mathbb{R}^2, \quad \varphi_1([x : y : 1]) = (x, y).$$

We denote each of these *coordinate identifications* by φ_i. Pairing the coordinate neighborhoods together with the coordinate identifications, we get *coordinate charts* identifying parts of $\mathbb{RP}^2$ with $\mathbb{R}^2$. Morally, we can treat $\mathbb{RP}^2$ locally like it is just $\mathbb{R}^2$, at least topologically.

Example 3. Consider the *coordinate cross* in $\mathbb{R}^2$:

$$C := \{(x, 0) \mid x \in \mathbb{R}\} \cup \{(0, y) \mid y \in \mathbb{R}\}.$$

Outside of $(0, 0)$, the coordinate cross has obvious identifications with $\mathbb{R}$. However, at the origin, the coordinate cross does not have any obvious identification, so we cannot pretend it is $\mathbb{R}$ locally.

So it is clear not every topological subspace of $\mathbb{R}^n$ locally looks like $\mathbb{R}^n$. Our main goal is to formalize what we mean for a space to locally look like Euclidean space, motivated by the above two examples.

Definition 6. A topological space M is *locally Euclidean of dimension* n if for every point $p \in M$, there is an open neighborhood U of p, an open subset $V \subseteq \mathbb{R}^n$, and a homeomorphism $\varphi : U \to V$. We call the pair (U, φ) a *chart*, where U is a *coordinate neighborhood* and φ is a *coordinate map.*

Exercise 3. Show that M is locally Euclidean of dimension n if and only if for every $p \in M$, there is an open neighborhood U of p and a homeomorphism $\varphi : U \to \mathbb{R}^n$.

Remark 4. In light of the previous exercise, in a chart (U, φ), we can write

$$\varphi = (\varphi_1, \ldots, \varphi_n),$$

where $\varphi_i : U \to \mathbb{R}$ are continuous maps. For the purposes of calculations, this lets us really treat U as if it were Euclidean space; just use φ_i as you would x_i in a real analysis class.

We now go through some examples of locally Euclidean spaces.

Exercise 4. Show that $\mathbb{RP}^2$ is locally Euclidean of dimension 2.

Example 4. Let
$$X := \{(x, x^{2/3}) \mid x \in \mathbb{R}\} \subseteq \mathbb{R}^2,$$
equipped with the subspace topology. We claim that X is locally Euclidean of dimension 1. To see this, define
$$\varphi : X \to \mathbb{R}, \quad \varphi(x, x^{2/3}) := x.$$
We first observe this map is continuous. First, recall that the topology on $\mathbb{R}^2$ is generated by the Euclidean norm $\|\cdot\|$. Suppose $(x_n, x_n^{2/3}) \to (x, x^{2/3})$ in this topology. Then for all $\varepsilon > 0$, there is an $N > 1$ such that for all $n \geq N$,
$$|x_n - x| \leq \|(x_n, x_n^{2/3}) - (x, x^{2/3})\| = \sqrt{|x_n - x|^2 + |x_n^{2/3} - x^{2/3}|^2} < \varepsilon.$$
Thus, we see that $\varphi(x_n, x_n^{2/3}) \to \varphi(x, x^{2/3})$. Now define
$$\psi : \mathbb{R} \to X, \quad \psi(x) := (x, x^{2/3}).$$
Clearly $\varphi \circ \psi(x) = x$ and $\psi \circ \varphi(x, x^{2/3}) = (x, x^{2/3})$, so these maps are inverses. Moreover, we see that if $x_n \to x$, then by continuity of the map $x \mapsto x^{2/3}$ we have $x_n^{2/3} \to x^{2/3}$. Thus, $\psi(x_n) \to \psi(x)$, and hence φ is a homeomorphism.

Exercise 5. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuous. We define the *graph* of f by
$$\mathrm{Graph}(f) := \{(x, f(x)) \mid x \in \mathbb{R}^n\} \subseteq \mathbb{R}^{n+m}.$$
Show that this is locally Euclidean of dimension n.

Example 5. Let C be the coordinate cross in $\mathbb{R}^2$. This is *not* locally Euclidean of any dimension. As observed earlier, C is locally Euclidean of dimension 1 outside of $(0,0)$. However, if it were locally Euclidean of dimension 1 at $(0,0)$, then we would have a neighborhood $U \subseteq C$ and a homeomorphism $\varphi : U \to \mathbb{R}$. Notice that this extends to a homeomorphism $\varphi : U \setminus \{(0,0)\} \to \mathbb{R} \setminus \{0\}$. But $U \setminus \{(0,0)\}$ has four connected components while $\mathbb{R} \setminus \{0\}$ has only two.

The definition of a topological manifold is technical. It arises from the fact that being locally Euclidean isn't necessarily good enough to turn all local problems into problems in Euclidean space. Indeed, there are many pathologies that arise from working with spaces which are just locally Euclidean. To avoid these pathologies, we introduce two more axioms which we would expect Euclidean-like spaces to satisfy.

Definition 7. A topological space M is a *topological manifold* of dimension n if it is Hausdorff, second countable, and locally Euclidean of dimension n.

Example 6. We give two examples of topological manifolds.

(i) Every open subset of $\mathbb{R}^n$ is a topological manifold.

(ii) Every graph of a continuous function is a topological manifold.

Exercise 6. Elaborate on the above example by showing that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous, then $\mathrm{Graph}(f)$ is Hausdorff and second countable.

We finish with two examples that are not either of the above.

Exercise 7. Show that $\mathbb{RP}^2$ is a topological manifold of dimension 2.

Example 7. Recall that the *unit-circle* is given by

$$\mathbb{S}^1 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\},$$

and is equipped with the subspace topology. There are four coordinate charts of interest:

$$\begin{cases} U_1 := \{(x,y) \in \mathbb{S}^1 \mid x < 0\}, \\ U_2 := \{(x,y) \in \mathbb{S}^1 \mid y < 0\}, \\ U_3 := \{(x,y) \in \mathbb{S}^1 \mid x > 0\}, \\ U_4 := \{(x,y) \in \mathbb{S}^1 \mid y > 0\}. \end{cases}$$

We then define maps

$$\begin{cases} \varphi_1 : U_1 \to (-1,1), & \varphi_1(x,y) := y, \\ \varphi_2 : U_2 \to (-1,1), & \varphi_2(x,y) := x, \\ \varphi_3 : U_3 \to (-1,1), & \varphi_3(x,y) := y, \\ \varphi_4 : U_4 \to (-1,1), & \varphi_4(x,y) := x. \end{cases}$$

We define their respective inverses by

$$\begin{cases} \psi_1 : (-1,1) \to U_1, & \psi_1(y) := (-\sqrt{1-y^2}, y), \\ \psi_2 : (-1,1) \to U_2, & \psi_2(x) := (x, -\sqrt{1-x^2}), \\ \psi_3 : (-1,1) \to U_3, & \psi_3(y) := (\sqrt{1-y^2}, y), \\ \psi_4 : (-1,1) \to U_4, & \psi_4(x) := (x, \sqrt{1-x^2}). \end{cases}$$

Exercise 8. Check that the above maps define charts on $\mathbb{S}^1$, so $\mathbb{S}^1$ is locally Euclidean of dimension 1. Also verify the remaining two axioms in order to show $\mathbb{S}^1$ is a topological manifold of dimension 1.

3. Lecture 2

Recall from last time that a *topological manifold* is a Hausdorff, second countable, *locally Euclidean* topological space, where locally Euclidean means that every point has a neighborhood homeomorphic to $\mathbb{R}^n$ for some constant $n \geq 1$.

Our goal now is to describe how one does calculus on these spaces. For example, if we take a curve $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = p$, we would say this curve is *differentiable* at a point $p \in M$ with respect to a chart (U, φ) if $\varphi \circ \gamma$ is differentiable at time $t = 0$. Taking a derivative of the curve then reduces to taking a derivative in these coordinates. However, this requires a choice of a chart; what happens if p lies in multiple charts? The aim of this lecture is to explore how to make such a definition of derivatives well-defined.

Definition 8. Let M be a topological manifold and let (U, φ) and (V, ψ) be two charts on M such that $U \cap V \neq \varnothing$. The *transition map* from (U, φ) to (V, ψ) is the map

$$\tau_{(U,\varphi),(V,\psi)} : \varphi(U \cap V) \to \psi(U \cap V), \quad \tau_{(U,\varphi),(V,\psi)} := \psi \circ \varphi^{-1}.$$

In particular, the transition map describes how derivatives may change between charts, if such a change of basis even makes sense.

Example 8. Let $M = \mathbb{R}$ be equipped with the Euclidean topology. Naturally, this is a topological manifold of dimension 1. We can equip this with two charts: $(\mathbb{R}, \varphi)$ and $(\mathbb{R}, \psi)$, where

$$\begin{cases} \varphi : \mathbb{R} \to \mathbb{R}, & \varphi(x) = x, \\ \psi : \mathbb{R} \to \mathbb{R}, & \psi(x) = x^3. \end{cases}$$

Note that

$$\tau_{(\mathbb{R},\psi),(\mathbb{R},\varphi)}(x) := \varphi(\psi^{-1}(x)) = x^{1/3}.$$

This map is *not* differentiable at $x = 0$.

One should interpret the previous example as saying that if one tries to measure the derivative of a function with these two different charts, one can get answers which are incompatible.

Example 9. Let (U_i, φ_i) be the charts on $\mathbb{S}^1$ from our earlier example. Observe

$$\tau_{(U_3,\varphi_3),(U_4,\varphi_4)} : (0,1) \to (0,1), \quad \tau_{(U_3,\varphi_3),(U_4,\varphi_4)}(y) := \sqrt{1 - y^2}.$$

This map is smooth.

One should interpret the previous example as saying that if one tries to measure the derivative of a function with these two different charts, one can gets compatible answers. Motivated by these two examples, we make the following definition.

Definition 9. Let M be a topological manifold. Two charts (U, φ) and (V, ψ) are *smoothly compatible* if the transition map from (U, φ) to (V, ψ) is smooth and the transition map from (V, ψ) to (U, φ) is smooth.

Exercise 9. Let (U_i, φ_i) be the charts on $\mathbb{S}^1$ from our earlier example. Show that all of the charts are smoothly compatible with one another.

We now want a family of charts which cover the manifold and which are all compatible with one another.

Definition 10. Let M be a locally Euclidean space. A *smooth atlas* is a family of charts $\mathcal{A} := \{(U_\alpha, \varphi_\alpha)\}$ which are smoothly compatible and which cover M, i.e.,

$$M = \bigcup_\alpha U_\alpha.$$

A chart (V, ψ) is *compatible* with the atlas $\mathcal{A}$ if it is compatible with all of the charts in the atlas. Two smooth atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ are *compatible* if $\mathcal{A}_1 \cup \mathcal{A}_2$ is a smooth atlas on M.

Exercise 10. Let $\mathbb{S}^1$ be the unit-circle from earlier. Consider

$$\begin{cases} V_1 := \{(\cos(\theta), \sin(\theta)) \mid \theta \in (-\pi, \pi)\}, \\ V_2 := \{(\cos(\theta), \sin(\theta)) \mid \theta \in (0, 2\pi)\}, \end{cases}$$

and let

$$\begin{cases} \varphi_1 : V_1 \to (-\pi, \pi), \quad \varphi_1(\cos(\theta), \sin(\theta)) := \theta, \\ \varphi_2 : V_2 \to (0, 2\pi), \quad \varphi_2(\cos(\theta), \sin(\theta)) := \theta. \end{cases}$$

(1) Calculate the transition maps. Is this a smooth atlas?

(2) If it is a smooth atlas, show that it is compatible with the smooth atlas from our previous example.

What happens if two charts are both compatible with an atlas?

Lemma 5. *Let $\mathcal{A}$ be a smooth atlas on a locally Euclidean space M. If (V, ψ) and (W, σ) are both compatible with $\mathcal{A}$, then they are compatible with each other.*

Proof. Let $p \in V \cap W$. Note that there must be some chart $(U, \varphi) \in \mathcal{A}$ such that $p \in U$. Hence:

$$\sigma \circ \psi^{-1} = (\sigma \circ \varphi^{-1}) \circ (\varphi \circ \psi^{-1}).$$

Note that equality here means equality on a common domain. We observe that this transition map is smooth since it is a composition of smooth functions. □

Definition 11. A smooth atlas $\mathcal{A}$ is *maximal* if it is not contained in another atlas. In other words, a smooth atlas $\mathcal{A}$ is maximal if for any other smooth atlas $\mathcal{A}'$, we have that $\mathcal{A} \subseteq \mathcal{A}'$ implies $\mathcal{A} = \mathcal{A}'$.

Let's first show that every smooth atlas is contained in a unique maximal smooth atlas.

Lemma 6. *Let M be a topological manifold and let $\mathcal{A}$ be a smooth atlas on M. There exists a unique maximal smooth atlas $\overline{\mathcal{A}}$ containing $\mathcal{A}$.*

Proof. Let $\mathcal{M}$ be the collection of all smooth atlases on M, and let

$$\mathcal{C}(\mathcal{A}) := \{\mathcal{A}' \in \mathcal{M} \mid \mathcal{A}' \cup \mathcal{A} \in \mathcal{M}\}.$$

Observe that $\mathcal{C}(\mathcal{A})$ consists of those atlases which are compatible with $\mathcal{A}$. One can use an argument similar to the previous lemma in order to show that

$$\overline{\mathcal{A}} := \bigcup_{\mathcal{A}' \in \mathcal{C}(\mathcal{A})} \mathcal{A}' \in \mathcal{M}.$$

We now claim that $\overline{\mathcal{A}}$ is maximal. By definition, note that $\mathcal{A} \in \mathcal{C}(\mathcal{A})$, and for every $\mathcal{A}' \in \mathcal{C}(\mathcal{A})$ we have $\mathcal{A}' \subseteq \overline{\mathcal{A}}$. If $\overline{\mathcal{A}} \subseteq \mathcal{B} \in \mathcal{M}$, then

$$\mathcal{A} \cup \mathcal{B} = \mathcal{B} \in \mathcal{M}.$$

This implies that $\mathcal{B} \in \mathcal{C}(\mathcal{A})$, so $\mathcal{B} \subseteq \overline{\mathcal{A}}$, proving that it is maximal.

Finally, we claim there is no other maximal smooth atlas containing $\mathcal{A}$. If $\mathcal{B}$ is a smooth atlas containing $\mathcal{A}$, then again $\mathcal{B} \in \mathcal{C}(\mathcal{A})$, which implies that $\mathcal{B} \subseteq \overline{\mathcal{A}}$. Thus, if $\mathcal{B}$ is maximal, then it is equal to $\overline{\mathcal{A}}$. □

Exercise 11. Prove that if $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A} \in \mathcal{M}$ are such that $\mathcal{A} \cup \mathcal{A}_1 \in \mathcal{M}$ and $\mathcal{A} \cup \mathcal{A}_2 \in \mathcal{M}$, then $\mathcal{A}_1 \cup \mathcal{A}_2 \in \mathcal{M}$.

In light of the above work, we see that maximal smooth atlases determine all pairwise compatible smooth atlases. This motivates the following.

Definition 12. Let M be a topological manifold. A *smooth structure* on M is a maximal smooth atlas. A *smooth manifold* is a topological manifold together with a smooth structure.

Example 10. The *Euclidean smooth structure* on $\mathbb{R}^n$ is the one corresponding to the atlas $\{(\mathbb{R}^n, \mathrm{Id})\}$.

Note that our earlier example shows that the smooth structure on $\mathbb{R}$ is not unique. One can modify the example in order to deduce the following.

Exercise 12. Show that there are uncountably many smooth structures on $\mathbb{R}$.

We finish by discussing a large class of smooth manifolds.

Definition 13. A subset $M \subseteq \mathbb{R}^n$ is a *smooth k-dimensional submanifold* of $\mathbb{R}^n$ if for every point $p \in M$, there exists an open neighborhood $U \subseteq \mathbb{R}^n$ of p, an open set $W \subseteq \mathbb{R}^k$, and a smooth map $F : W \to U \cap M$ which is a homeomorphism and such that $F^{-1} : U \cap M \to W$ is also smooth.

Warning: While the naming convention may be confusing, note that we are *not* saying that all smooth manifolds inside of $\mathbb{R}^n$ are smooth submanifolds. For example, $\mathbb{R}$ equipped with the smooth structure associated to the atlas $\{(\mathbb{R}, x \mapsto x^3)\}$ is not a smooth submanifold of $\mathbb{R}$, even though it is a smooth manifold.

Example 11. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth function. Apriori, we know that $\mathrm{Graph}(f)$ is a topological manifold with one chart. Since this chart is defined via a smooth function (with respect to the Euclidean smooth structure), it defines a smooth submanifold.

Example 12. All open subsets of $\mathbb{R}^n$ are smooth submanifolds.

Exercise 13. Show that $\mathbb{S}^1$ with the smooth structure we defined earlier is a smooth submanifold of $\mathbb{R}^2$.

Example 13. Denote the collection of $n \times n$ matrices with real coefficients by $M_n(\mathbb{R})$. We will adopt the notation

$$(a_{ij})_{1 \le i,j \le n} = \begin{pmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}.$$

Note that there is a natural identification of this space with $\mathbb{R}^{n^2}$ by

$$M_n(\mathbb{R}) \ni (a_{ij})_{1 \le i,j \le n} \mapsto (a_{11}, \dots, a_{1n}, a_{2n}, \dots, a_{nn}) \in \mathbb{R}^{n^2}.$$

In particular, $M_n(\mathbb{R})$ is a smooth manifold with one chart. We define the *general linear group* to be the set

$$\mathrm{GL}(n, \mathbb{R}) := \{A \in M_n(\mathbb{R}) \mid \det(A) \neq 0\}.$$

Recall that $\det : M_n(\mathbb{R}) \to \mathbb{R}$ is a continuous map, and so

$$\mathrm{GL}(n, \mathbb{R}) = \det{}^{-1}((-\infty, 0) \cup (0, \infty)),$$

hence $\mathrm{GL}(n, \mathbb{R})$ is an open subset of $M_n(\mathbb{R})$. Through our above identification, $\mathrm{GL}(n, \mathbb{R})$ can be viewed as a smooth submanifold of $\mathbb{R}^{n^2}$.

Exercise 14. Calculate the dimension of $\mathrm{GL}(n, \mathbb{R})$ as a smooth submanifold.
Hint: There is a more general fact that you should use here.