Real Analysis

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Contents

Chapter 1

Prerequisites

1.1 Sets, Functions, and $\mathbb{R}$

We start with the very vague definition of a set (for a more formal definition, look into ZFC set theory).

Definition (Set)**.** A set is a collection of distinct objects.

Example 1.1. The collection $\{a, b, c\}$ is a set.

Definition (Empty Set)**.** The empty set, denoted $\varnothing$, is the set with no objects.

Sets are very intuitive by design. They are what we naturally think about things. Objects must be members of sets, and generally this is denoted by $\in$.

Definition (Subset)**.** A subset of a set X is a collection of objects C such that for all $x \in C$, $x \in X$. We denote this by $C \subset X$. If C could be equal to X, we denote it by $C \subseteq X$.

Example 1.2. If $X = \{a, b, c\}$, $C = \{a, b\}$, then $C \subset X$ (this is strict).

We can do some basic operations on sets, such as taking the product of sets, taking the union of sets, taking the intersection of sets, and subtracting sets.

Definition (Product)**.** The (Cartesian) product of two sets X and Y is the set

$$X \times Y = \{(a, b) \ : \ a \in X, b \in Y\}.$$

Definition (Union)**.** The union of two sets X and Y is the set

$$X \cup Y = \{a \ : \ a \in X \text{ or } a \in Y\}.$$

Definition (Intersection)**.** The intersection of two sets X and Y is the set

$$X \cap Y = \{a \ : \ a \in X \text{ and } a \in Y\}.$$

Definition (Set minus)**.** The difference or set minus between two sets X and Y is

$$Y \backslash X = \{y \in Y \ : \ y \notin x\}.$$

Notice we get the relation

$$X \cap Y \subseteq X, Y \subseteq X \cup Y.$$

Definition (Powerset)**.** The powerset of a set X is denoted $P(X)$, and it is the set of all subsets of X. Notationally,

$$P(X) = \{C \ : \ C \subseteq X\}.$$

Example 1.3. If $X = \{a, b, c\}$, $Y = \{b, c, d\}$, then

$$X \cap Y = \{b, c\},$$

$$X \cup Y = \{a, b, c, d\},$$

$$X \times Y = \{(a, b), (a, c), (a, d), (b, b), (b, c), (b, d), (c, b), (c, c), (c, d)\},$$

$$P(X) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\} = X\}.$$

We want to compare sets, and one way to do so is a function.

Definition (Function)**.** For two sets X, Y, a function $f : X \to Y$ is an assignment of objects from X to objects in Y.

More formally, we want our functions to be well-defined.

Definition (Well-Defined)**.** We say that a function $f : X \to Y$ is well-defined if for $y_1, y_2 \in Y$, $f(x) = y_1$ and $f(x) = y_2$ implies $y_1 = y_2$. Loosely speaking, if $f(x)$ maps to only one thing.

Example 1.4. If $X = \mathbb{R}$, $Y = \mathbb{R}$, then $f(x) = x$ is a function.

Definition (Injective)**.** A function $f : X \to Y$ is injective if $f(x) = f(y)$ implies $x = y$.

Definition (Surjective)**.** A function $f : X \to Y$ is surjective if for all $y \in Y$, there exists $x \in X$ such that $f(x) = y$.

Example 1.5 (Non-example)**.** The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is not surjective (it does not hit negative numbers). Moreover, it is not injective; $f(-1) = 1$, $f(1) = 1$, but $-1 \neq 1$. However, restricting the codomain to postive real numbers, we have $f : \mathbb{R} \to \mathbb{R}_{\geqslant 0}$ is surjective, since for any $x \in \mathbb{R}_{\geqslant 0}$, we can take $\sqrt{x} \in \mathbb{R}$ so that $f(\sqrt{x}) = x$.

Definition (Bijective)**.** We say that a function $f : X \to Y$ is bijective if it is injective and surjective.

One thing we will want to do is combine functions, through composition.

Definition (Composition)**.** If $f : X \to Y$, $g : Y \to Z$, then the function $g \circ f : X \to Z$ is the function which takes $x \in X$ to $g(f(x))$.

Proposition 1.1. If $f : X \to Y$ is injective, $g : Y \to Z$ is injective, then $g \circ f$ is injective.

Proof. If g, f are both injective, then $g(f(x)) = g(f(y))$ implies $f(x) = f(y)$ implies $x = y$, so their composition is injective. **Q.E.D**

Proposition 1.2. If $f : X \to Y$ and $g : Y \to Z$ are both surjective, then $g \circ f$ is surjective.

Proof. If g is surjective, then for all $z \in Z$ there is some $y \in Y$ so that $g(y) = z$. Since f is surjective, for all $y \in Y$ there is some $x \in X$ so that $f(x) = y$. Hence, $g(f(x)) = g(y) = z$. **Q.E.D**

Corollary 1.0.1. If $f : X \to Y$ and $g : Y \to Z$ are both bijective, then $g \circ f$ is bijective.

Example 1.6. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$ is injective and surjective, and so it is bijective.

Definition (Countable)**.** We say that a set X is countable if there is a bijective function between it and the natural numbers, $\mathbb{N} = \{1, 2, 3, \ldots\}$.

Remark 1. In general, we would like to construct our examples from prior examples. There is no real way to construct the natural numbers though, so we will just assume their existence.

Example 1.7. We have that $X = \{2k, k \geqslant 1\}$ (the set of even positive numbers) is countable. We construct the bijection $f : X \to \mathbb{N}$ via $f(x) = x/2$. Then it is surjective, since for any $n \in \mathbb{N}$ I simply take $2n \in X$ so that $f(2n) = n$, and it is injective since $f(x) = f(y) = n$ implies $x/2 = y/2 = n$, or $x = y = 2n$.

Example 1.8. Less trivially, the set $X = \{p : \text{ p is prime}\}$ is countable as well.

One thing we may want to do is see which sets are "essentially" the same. To do so, we introduce something called an equivalence relation.

Definition (Binary Relation)**.** A binary relation on a set X is a subset R of the product $A \times A$. A binary relation between two sets X, Y is a subset of the product $X \times Y$. We say that $(x, y) \in X \times Y$ are related if $(x, y) \in R \subseteq X \times Y$.

Definition (Equivalence Relation)**.** An equivalence relation is a binary relation $R \subseteq X \times X$ which satisfies three properties:

i. (Reflexive) For all $x \in X$, $(x, x) \in R$.

ii. (Symmetric) For all $x, y \in X$, $(x, y) \in R$ implies $(y, x) \in R$.

iii. (Transitive) For all $x, y, z \in X$, $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$.

Example 1.9. Taking $X = \{a, b, c\}$, we have

$$X \times X = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}.$$

Taking a subset $R = \{(a, a), (a, b), (b, a), (b, b), (c, c)\} \subseteq X \times X$ gives us an equivalence relation (Exercise: Check the properties).

Remark 2. Equivalence relations are denoted generally with either R or $\sim$. We denote two elements x and y being related by either xRy or $x \sim y$.

One thing we would like to do (and will like to do often) is quotient things. In laymen terms, to quotient is to divide up our set based on equivalence classes. If you had a collection of coins, for example, their equivalence classes could be their amounts, and to divide them up is to put them in piles corresponding to their class. The more rigorous definition is given below.

Definition (Equivalence Class)**.** Given a set S and an equivalence relation R, we define an equivalence class to be

$$\bar{x} = [x] = \{a \in S \ : \ aRx\}.$$

Definition (Quotient Set)**.** Given a set S and an equivalence relation R, we define the quotient set to be the set of all equivalence classes. That is,

$$S/R = \{[x] \ : \ x \in R\}.$$

Remark 3. Instead of quotient, some say modulo. This comes from a very classical example, given below.

Example 1.10 (Modulo Integers)**.** Take the set of integers $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$, which we construct by groupifying the natural numbers (see next section). By the division algorithm (see future sections), we know that we can write all integers as

$$x = ay + r.$$

So, we create an equivalence relation based on this r, the remainder of a number. Notationally, we denote this by

$$x \pmod{y} = r.$$

Then we have

$$R = \{(x, y) \ : \ x \equiv y \pmod{n}\}$$

is an equivalence relation (it is a fun exercise to check this, although we'll do it later on). The quotient set is then

$$\mathbb{Z}/R = \{[0], [1], \ldots, [n-1]\}.$$

Quotienting is a very important concept, and will come up often. As a result, one should take the time to explore what it really means to be an equivalence relation and to quotient by an equivalence relation.

Remark 4. The astute reader may also notice that this gives us a very natural function from our original set S to our quotient set S/R. This function is just sending an element to its equivalence class. By construction, this function is surjective, so this gives us a way to relate this objects back to our original objects.

One thing we will want to do is explore functions between sets. One may ask, for example, how many functions exist between sets, how many injective functions exist between sets, and how many surjective functions exist between sets? We explore some of these questions now.

Definition (Cardinality)**.** We define the cardinality of a set X to be the number of elements in it. It is generally denoted by $|X|$.

Proposition 1.3. The number of functions between two finite sets X and Y with domain in X and codomain in Y is $|Y|^{|X|}$.

Proof. For each element $x \in X$, we have that it could map onto $|Y|$ different candidates. Hence, we get

$$|Y| \cdot |Y| \cdots |Y| = |Y|^{|X|}.$$

Q.E.D

Proposition 1.4. Given that $|X| < |Y| < \infty$, the number of injective functions between X and Y with domain in X and codomain in Y is

$$|Y| \cdot (|Y| - 1) \cdot (|Y| - 2) \cdots (|Y| - |X| + 1).$$

Proof. We first select where $x \in X$ is going to go. We have $|Y|$ possible options for this. Now, we decide where $x' \in X \backslash x$ is going to go. Since the function is injective, it cannot go to where the first element went. There are $|Y| - 1$ options for this. Continuing down the line, we get the above equation. **Q.E.D**

For surjective functions, we need something a bit more technical, which is the inclusion-exclusion principle.

Theorem 1.1 (Inclusion-Exclusion Principle)**.** For finite sets $A_1, \ldots, A_n$, we have

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{i=1}^{n} |A_i| - \sum_{1 \leqslant i < j \leqslant n}^{n} |A_i \cap A_j| + \cdots + (-1)^n \left| \bigcap_{i=1}^{n} A_i \right|.$$

Proposition 1.5. Given that $|Y| < |X| < \infty$, the number of surjective functions between X and Y with domain in X and codomain in Y is

$$\sum_{i=1}^{|X|} (-1)^{|X|} \binom{|X|}{i} (|X| - i)^{|Y|}$$

From intuition and from what was discussed, we know that $\mathbb{Q} = \{p/q : p \in \mathbb{Z}, q \in \mathbb{Z}\backslash\{0\}\} \subseteq \mathbb{R}$. However, one may wonder whether $\mathbb{R} \subseteq \mathbb{Q}$; that is, do we have equality? The ancient Greeks believed this; they thought that every number was representable by rational numbers. However, the reality is not as clean as we would like. It turns out, in fact, that we have a strict inequality $\mathbb{Q} \subsetneq \mathbb{R}$. To see this, we propose the following theorem;

Theorem 1.2. The solution to $x^2 = 2$, that is, $\pm\sqrt{2}$, is not a rational number.

Before proving this, I'd like to note that this is a common path of reaching the reals (and furthermore, the complex numbers). We try to solve polynomials with coefficients in our field of choice, and see if the roots remain in the field. If not, it's clear we need to expand our field to append these kinds of roots, and then we should check that this remains a field. But we'll go more into this later.

Proof. We first propose a lemma, based on the **Fundamental Theorem of Arithmetic** (which I will not prove quite yet, as it's an involved proof).

Lemma 1.1. If a prime q divides an integer p^2, then q divides p.

I'll leave this as an exercise (we discussed this in person). Using this, though, we will prove that $\sqrt{2}$ is irrational through a proof by contradiction. Assume

$$\sqrt{2} = \frac{p}{q},$$

where $p \in \mathbb{Z}$ and $q \in \mathbb{Z}\backslash\{0\}$. Furthermore, assume that p and q do not share any factors; that is, this is a reduced representation. Doing some algebra, we have

$$\frac{p^2}{q^2} = 2 \implies p^2 = 2q^2.$$

This implies, by definition, that 2 divides p^2. By our lemma above, we get 2 divides p. Thus, we may write $p = 2k$, where $k \in \mathbb{Z}$. Thus, we rewrite our equality as

$$(2k)^2 = 4k^2 = 2q^2 \implies 2k^2 = q^2.$$

This then gives us that 2 divides q^2, and so 2 divides q. However, this means that p and q share a factor of 2, and so we have reached a contradiction. **Q.E.D**

To finish this chapter, let's discuss a theorem called **Cantor's theorem**.

Theorem 1.3. For any set X, we have $X \subsetneq \mathcal{P}(X)$.

This theorem, while intuitively true, gives us some very big issues with our definition of set. For example, we could declare $\mathfrak{U}$ to be the universal set; that is, the set of all sets. But this gives us an issue; declaring $\mathfrak{U}$ as so would imply that $\mathcal{P}(\mathfrak{U}) \subseteq \mathfrak{U}$, but by Cantor's theorem we have that this cannot be true. It's clear, then, that we need to be very careful with our definition of sets if we were to be completely formal. We will not be completely formal here, as this is not the purpose of these notes, but I encourage you to look into this matter more.

1.2 Quantifiers, Convergence

Quantifiers are a way of converting phrases into what is called formal logic. Formal logic allows us to manipulate things in such a way that we may reach tautologies, or truths. This is clearly an important factor of mathematics, and so we see that these tools are important moving forward. I will warn you that, while useful at first and for shorthand notes, using quantifiers in more official proofs is discouraged, as things can get very messy very fast.

Definition. We have the following quantifiers:

1. We denote the quanitifier "for all" by $\forall$.
2. We denote the quanitifier "there exists" by $\exists$.
3. We denote the quanitifer "such that" by $\ni$.

These will be the important ones to us moving forward. We also sometimes care about the negation, denoted by $\neg$. This is the same as taking the opposite statement. Note that the negation of $\forall$ is $\exists$ and vice versa. These are mostly intuitive, and so I'll skim over them for now, but it's important to play with these things and see how they work.

Moving into real analysis, we care about things called sequences.

Definition. A sequence in $\mathbb{R}$ is a collection of objects $\{a_0, a_1, \ldots, a_n, \ldots\} = \{a_i\}_{n \geqslant 0} = \{a_i\}$ such that $a_i \in \mathbb{R}$ for all i.

A lot of real analysis is studying these sequences. Sequences are important in approximating things, for example, we could approximate functions through Taylor's theorem, which gives us a sequence of coefficients, or we could approximate numbers by their closest rational number, or more. Approximation is at the heart of analysis and mathematics, and so it's important to hold this sort of ideology in your mind moving forward (though I may not be very explicit with how these things could be used to approximate things).

In order for an approximation to be accurate, we need it to be *extremely* close to our goal number. So, if $\{a_i\}$ is approximating a number, let's say L, then we want it to be very close to L. However, this is an approximation; so it may be that it will never reach L. We thus need a condition that measures *extremely close*. One such way is as follows;

Definition. Let $\{a_i\}$ be a sequence in $\mathbb{R}$. Then we say that $\{a_i\}$ converges to a number L if, for all $\epsilon > 0$, there exists an n_0 such that for all $n \geqslant n_0$ we have

$$|a_n - L| < \epsilon.$$

We write $\{a_n\} \to L$ or $a_n \to L$.

What this definition is saying is that, for any ball centered at L with radius ϵ, I can find a point in my sequence, which is n_0, so that all of my terms after and including n_0 will be within this ball. Since I can do this for all $\epsilon > 0$, it is clear that this will be a very good approximation of my number L. It's good to meditate on this for time and figure out for yourself why this is true.

Now, we could just simply say that a sequence diverges if it does not converge (which is a valid and complete definition), but for practice let's negate the statement I just gave.

Definition. Let $\{a_i\}$ be a sequence in $\mathbb{R}$. Then we say that $\{a_i\}$ diverges if there exists an $\epsilon > 0$ and an n_0 so that for all $n \geqslant n_0$, we have

$$|a_n - L| > \epsilon.$$

To translate this back to the ball example, there exists a point in the sequence where I will never be inside the ball. So while I may jump in and out of the ball at the beginning, as I go towards infinity I will eventually just not be in the ball at all.

It's worth noting here that the sequence need not be in $\mathbb{R}$; there is nothing intrinsic to the reals and sequences. However, we will have to modify the definition of convergence if we leave the reals, as the definition we gave requires us to be in what is called a metric space. We will discuss more on metric spaces later.

Let's explore one example of convergence.

Example 1.11. The sequence $\{1/n\}_{n \geqslant 1}$ converges to 0 in $\mathbb{R}$.

Proof. We will use what is called the **Archimedean Property** here, or in other words the fact that $\mathbb{N}$ is unbounded. We need to show that for all $\epsilon > 0$, there exists an n_0 so that for all $n \geqslant n_0$ we have

$$\left|\frac{1}{n}\right| = \frac{1}{n} < \epsilon.$$

Assume for contradiction that there is no such point for an ϵ. Then we have

$$\frac{1}{n} > \epsilon.$$

Using algebra, we rewrite this as

$$\frac{1}{\epsilon} > n.$$

However, this implies that there is an upper bound, $\frac{1}{\epsilon}$ to the natural numbers. This is a contradiction, and so we have that there must be some point where $1/n < \epsilon$. **Q.E.D**

One important way of looking into convergence of sequences is looking at the subsequences. Let's define what we mean by a subsequence.

Definition. Let $\{a_k\}$ be a sequence of real valued numbers. Then a subsequence $\{a_{n_k}\}$ is a subcollection of these numbers such that $n_k < n_{k+1}$ for all k.

Example 1.12. Let $\{1, 2, 3, 4, 5, \ldots\}$ be a sequence of integers. Then a subsequence would be the sequence of even integers, $\{2, 4, 6, 8, \ldots\}$. A non-example of a subsequence would be $\{2, 6, 4, 8, \ldots\}$, since we have that 4 comes before 6 in our original sequence. An important thing to note is that the subsequence preserves the order of the original sequence.

Now let's see how subsequences relate to convergence.

Theorem 1.4. We have $\{a_n\} \to L$ if and only if all subsequences converge as well.

Proof. Let's prove the implication. If $a_n \to L$, then we have for all $\epsilon > 0$ there exists a point n_0 so that for all $n \geqslant n_0$,

$$|a_n - L| < \epsilon.$$

Now, let a_{i_n} be a subsequence of a_n. Then we have $i_n \geqslant n$ for all i. So if we have an n_0, then we have for all $n \geqslant n_0$,

$$|a_{i_n} - L| \leqslant |a_n - L| < \epsilon.$$

The converse is trivial; the sequence is a subsequence of itself. **Q.E.D**

This seems to be tricky; in order to show that $\{a_n\}$ converges using this theorem, we would have to show that **all** subsequences converge, which is not a trivial matter. However, let's look at the negation of this statement (that is, the contrapositive).

Theorem 1.5. We have $\{a_n\}$ does **not** converge to L if there exists a subsequence which does not converge to L.

Example 1.13. The sequence $(-1)^n$ does not converge in $\mathbb{R}$.

Proof. We're going to use the prior theorem to prove this. Notice that taking the subsequence $(-1)^n$ for n even gives us a sequence converging to 1. However, taking the subsequence $(-1)^n$ for n odd gives us a sequence converging to -1. Since these do not agree, we do not have convergence. **Q.E.D**

Before finishing, let's talk about some properties of sequences.

Definition. We say a sequence is bounded if we have $|a_n| \leqslant M$ for all n.

Definition. We have that a sequence is increasing (decreasing respectively) if $a_n \leqslant a_{n+1}$ for all n ($a_n > a_{n+1}$ for all n respectively). We say that a sequence is strictly increasing (strictly decreasing respectively) if $a_n < a_{n+1}$ for all n ($a_n > a_{n+1}$ for all n respectively). We say that a sequence is monotonic if it is increasing or decreasing. We generally specify whether it is monotonically increasing or monotonically decreasing if it is not clear from context.

Example 1.14. The sequence $\{(-1)^n\}_{n=0}^{\infty}$ is bounded, as we have $|(-1)^n| = 1$ for all n. It is not, however, monotonic.

Example 1.15. The sequence $\{1/n\}_{n=1}^{\infty}$ is monotonically decreasing. This is because

$$\frac{1}{n} > \frac{1}{n+1} \ \forall n.$$

Moreover, we get that this is strictly decreasing.

We can couple these definitions together to get a powerful statement.

Theorem 1.6. (Monotone Convergence Theorem) Let $\{a_n\}$ be a sequence of real valued numbers. If $\{a_n\}$ is monotonic and bounded, then we have that it converges to it's bound.

Proof. Let's examine the case where $\{a_n\}$ is monotonically increasing and bounded (the case of monotonically decreasing and bounded is analogous). Then we have

$$|a_n| \leqslant M$$

for all n. Assume that M is the **least upper bound**. That is, for all $\epsilon > 0$, we have that there exists an n_0 so that

$$a_{n_0} > M - \epsilon.$$

We may find such an M by the **least upper bound property** of the reals (we will talk more about this when we talk about Cauchy sequences). Since the sequence is increasing, we get that this holds for all $n \geqslant n_0$ as well, so that we have

$$a_n > M - \epsilon \ \forall n.$$

Using some algebra, we get

$$\epsilon > |M - a_n|.$$

Hence, we get that M is the limit. **Q.E.D**

Lemma 1.2. The following are properties of limits of sequences:

(i) For sequences a_n, b_n (we drop the curly braces since this is clearly a sequence from context) which converge, we have

$$\lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = \lim_{n\to\infty} (a_n \pm b_n).$$

(ii) Let c be a constant and a_n a sequence that converges. Then we have

$$c \lim_{n\to\infty} a_n = \lim_{n\to\infty} ca_n.$$

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