DEFORMATIVE MAGNETIC MARKED LENGTH SPECTRUM RIGIDITY – PURDUE UNIVERSITY

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Abstract. Given a closed surface with negative curvature, any closed curve on the surface which is not null-homotopic can be continuously deformed to a unique closed geodesic. The marked length spectrum is a function which takes a closed curve and returns the length of the corresponding geodesic. It was shown by Guillemin and Kazhdan that if two negatively curved metrics on a closed surface can be connected by a path of negatively curved metrics along which the marked length spectrum is constant, then the metrics are isometric. In this talk, I will present a generalization of this theorem to the setting of magnetic flows on a closed surface with negative magnetic curvature.

Let (M, g) be a closed Riemannian manifold. Given a closed two form $\sigma \in \Omega^2(M)$, we say that a smooth curve γ is a **magnetic geodesic** if it satisfies the differential equation

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \Omega(\dot{\gamma}),$$

where $\Omega : TM \to TM$ is the Lorentz operator associated to σ:

$$g(v, \Omega(w)) = \sigma(v, w).$$

We denote the magnetic flow on TM by $\varphi_t^\sigma : TM \to TM$. Magnetic flows have a rich history in dynamical systems, dating back to Anosov and Sinai in the 60's. Recent interest in these dynamical systems comes from their relation to geodesic flows. By setting $\sigma \equiv 0$, we recover the geodesic flow, but we see that varying σ gives rise to Hamiltonian flows which have different behavior compared to the geodesic flow. One easy way to see this difference is in reversibility of the system – the geodesic flow is reversible, in the sense that if γ is a geodesic then $t \mapsto \gamma(-t)$ is also a geodesic. However, this is not true in general for magnetic systems.

Even with these differences, a remarkable number of results that hold for geodesic flows also hold for magnetic flows after appropriate adjustments. For example, if M is an orientable surface, then one can derive the magnetic Jacobi equation along a unit speed magnetic geodesic $\dot{\gamma}$ relative to the frame $\{\dot{\gamma}, \dot{\gamma}^\perp\}$. Recall that if M is a surface then we have $\sigma = b\mathrm{Vol}$; we call b the **magnetic intensity** of the system. If we have a Jacobi field $J(t) = x\dot{\gamma}(t) + y\dot{\gamma}^\perp(t)$, then one can derive

$$\begin{cases} x''(t) = (yb)' \\ y''(t) + (K^g(t) - db(\dot{\gamma}^\perp) + b^2)y = 0, \end{cases}$$

where K^g is the Gaussian curvature. Writing $K^{g,b}(x, v) = K^g(x) - db(v^\perp) + b^2(x)$ and calling this the **magnetic curvature**, we have that this term plays a role similar to the curvature term in Riemannian geometry. One similarity is that if $K^{g,b} < 0$, then the flow on the unit tangent bundle $\varphi_t^\sigma : S^gM \to S^gM$ is **Anosov**, meaning there is a splitting of the tangent bundle of the unit tangent bundle $T_{(x,v)}S^gM = E^s(x, v) \oplus \mathbb{R}F(x, v) \oplus E^u(x, v)$ such that

1. $F(x, v) = \frac{d}{dt}\big|_{t=0}\varphi_t^\sigma(x, v) \neq \{0\}$,
2. there exists $C, \lambda > 0$ so that for every $\xi^s \in E^s(x, v)$, $\xi^u \in E^u(x, v)$, and $t \geqslant 0$ we have

$$\|d\varphi_t^\sigma(\xi^s)\| \leqslant Ce^{-\lambda t}\|\xi^s\| \text{ and } \|d\varphi_{-t}^\sigma(\xi^u)\| \leqslant Ce^{-\lambda t}\|\xi^u\|.$$

The goal of today's talk is to discuss a generalization of Guillemin and Kazhdan's isospectral rigidity result on surfaces. It is well-known that for a closed manifold (M, g) if the sectional curvature is negative, then in every non-trivial free homotopy class there exists a unique closed geodesic. The **marked length spectrum** is the function MLS_g on the space of free homotopy classes which returns the length of the unique closed geodesic in the free homotopy class. It was conjectured by Burns and Katok in the mid 80's that this function determines the geometry of the the manifold, in the sense that $\mathrm{MLS}_g = \mathrm{MLS}_{g'}$ if and only if g and g' are isometric. This conjecture was motivated partially by the deformative case, which was proved by Guillemin and Kazhdan in 1980.

Theorem 1 (Guillemin and Kazhdan, '80)**.** Let M be a closed orientable surface, and let $\{g_s\}$ be a smooth family of metrics on M such that $K^{g_s} < 0$ for each s. If $\mathrm{MLS}_{g_s} = \mathrm{MLS}_{g_0}$ for every s, then there exist a smooth family of diffeomorphisms $f_s : M \to M$ such that $f_s^*(g_s) = g_0$.

We'll briefly mention some of the history behind this conjecture.

- In '90, Croke and Otal (seperately) solved the conjecture in the case where M is a surface.
- In the late '90's, Besson-Courtois-Gallot and Ursula showed the conjecture holds if the manifold is locally symmetric.
- Recently, Guillarmou and Lefeuvre showed the conjecture holds if the metrics are "close" (where close is with respect to some fine topology).

In full generality, the conjecture is still open.

If one assumes that M is a surface and $K^{g,b} < 0$ everywhere, then we still have that inside of every non-trivial free homotopy class there is a unique closed magnetic geodesic. With this in mind, we define the marked length spectrum function in the same way, and we can ask the same question as Burns and Katok – namely, if $\mathrm{MLS}_{g,b} = \mathrm{MLS}_{g',b'}$, does this imply the existence of a diffeomorphism $f : M \to M$ such that $f^*(g') = g$ **and** $f^*(b') = b$? Some results are known in this direction – for example, Grognet in the '90's showed that $\mathrm{MLS}_{g,b} = \mathrm{MLS}_{g',0}$ if and only if g and g' are isometric and $b \equiv 0$. Beyond this, however, not much has been explored. Recently, I discovered that you can generalize Guillemin and Kazhdan to the magnetic scenario.

Theorem 2 (Marshall Reber, '22)**.** Let M be a closed oriented surface, and let $\{(g_s, b_s)\}$ be a smooth family of magnetic systems on M such that $K^{g_s,b_s} < 0$ for each s. If $\mathrm{MLS}_{g_s,b_s} = \mathrm{MLS}_{g_0,b_0}$ for every s, then there exists a smooth family of diffeomorphisms $f_s : M \to M$ such that $f_s^*(g_s) = g_0$ and $f_s^*(b_s) = b_0$.

The proof of this is somewhat technical, but I'll discuss the steps that go into it. First, let

$$H_s(x, v) := \frac{1}{2} g_s(v, v), \quad \beta_s := \frac{d}{ds} H_s.$$

The idea is to show that there is a 1-form δ_s such that, if X_s is the infinitesmal generator of the geodesic flow for the metric g_s, then $X_s(\delta_s) = \beta_s$. Let Z_s be the vector field dual to δ_s, that is,

$$g_s(Z_s(x), v) = \delta_s(v) \text{ for } (x, v) \in TM.$$

Notice this gives us a time-dependent vector field Z_s. Observe that given $(x, v) \in TM$ and γ_v the corresponding geodesic for (x, v), we have

$$\frac{d}{ds} H_s(v) = \beta_s(v, v) = X_s(\delta_s)(v) = \frac{d}{dt}\Big|_{t=0} \delta_s(\dot{\gamma}_v(t)) = \frac{d}{dt}\Big|_{t=0} g_s(Z_s(\gamma(t)), \dot{\gamma}(t)) = g_s(\nabla_v Z_s(x), v).$$

Next, using ODE's, we can solve the differential equation

$$\frac{d}{ds} f_s(p) = Z_s(f_s(p)), \quad f_0(p) = p$$

in order to find a smooth family of diffeomorphisms $f_s : M \to M$. Consider the metrics $g'_s := f_s^*(g_0)$ and let

$$H'_s(x,v) := \frac{1}{2} g'_s(v,v), \quad \beta'_s := \frac{d}{ds} H'_s.$$

For $(x,v) \in TM$ and γ_v the corresponding geodesic for (x,v), we have

$$\begin{aligned}\frac{d}{ds} H'_s(v) = \frac{1}{2}\frac{d}{ds} g_0(df_s(v), df_s(v)) &= \frac{1}{2}\frac{d}{ds} g_0\left(\frac{d}{dt}\Big|_{t=0} f_s \circ \gamma_v(t), \frac{d}{dt}\Big|_{t=0} f_s \circ \gamma_v(t)\right) \\ &= g_0\left(\nabla_{d/ds} \frac{d}{dt}\Big|_{t=0} f_s \circ \gamma_v(t), \frac{d}{dt}\Big|_{t=0} f_s \circ \gamma_v(t)\right) \\ &= g_0(\nabla_{df_s(v)} Z_s(f(x)), df_s(v)) \\ &= g'_s(\nabla_v Z_s(x), v).\end{aligned}$$

Thus we see H_s and H'_s both satisfy the same differential equation with $H_0 = H'_0$. As a result, they must be equal, so after a polarization argument we have that $g_s = f_s^*(g_s)$.

The missing ingredient in the above is showing the existence of the 1-form. This is crucial for Guillemin and Kazhdan, and they prove this using a Fourier analysis argument. We can adopt the same framework for the magnetic setting with some alterations. Fixing a magnetic system (g,b), let $\{X, X^\perp, V\}$ be Cartan's moving frame on S^gM. Recall that X is the infinitesmal generator of the geodesic flow, while V is the infinitesmal generator of the rotation flow. Using Stone's theorem, we can consider an operator $-iV$ on $L^2(S^gM, \mathbb{C})$ which is densely defined. We consider the decomposition with respect to this operator:

$$L^2(S^gM,\mathbb{C}) = \bigoplus_k H_k, \text{ where } H_k := \{f \in L^2(S^gM,\mathbb{C}) \mid -iVf = kf\}.$$

Every $f \in L^2(S^gM, \mathbb{C})$ can be expressed as

$$f = \sum_k f_k \text{ where } f_k \in H_k.$$

We'll be focused on the smooth function case, so we'll let $\Omega_k = H_k \cap C^\infty(S^gM,\mathbb{C})$ and study function with respect to this decomposition. I showed in the same paper that we have the following "magnetic Carleman estimate."

Theorem 3 (Marshall Reber, '22)**.** Assume that (g,b) generates a magnetic flow with infinitesmal generator F. Let $u \in C^\infty(S^gM,\mathbb{C})$ have finite degree. If $-2b \leqslant K^{g,b} \leqslant -2a < 0$ for some $a, b > 0$, then for every $\sigma > 0$ there is a sequence of real numbers (γ_k) such that for any $N \geqslant 1$ we have

$$\sum_{|k|\geqslant N}^{\infty} \gamma_k^2 \|u_k\|^2 \leqslant \frac{2}{ae^\sigma} \sum_{|k|\geqslant N+1}^{\infty} \gamma_k^2 \|(Fu)_k\|^2.$$

Remark 1. Observe that one difference between this and the usual Carleman estimate is that we have to assume that u has finite degree. Much of the technical details are devoted to fighting this issue.

Now, the family $\{\beta_s\}$ are functions living in $C^\infty(S^gM,\mathbb{C})$, and so can be written in terms of their Fourier components. In particular, it is not too hard to see that, since β_s can be viewed as a real-valued symmetric 2-tensor, we have that β_s has degree 2 and can be written as

$$\beta_s = \beta_{-2}^s + \beta_0^s + \beta_2^s, \text{ where } \overline{\beta_{-2}^s} = \beta_2^s.$$

The trick here is to show that

$$\int_\gamma \beta_s = 0 \text{ for every magnetic geodesic } \gamma_s \text{ for the system } (g_s, b_s).$$

Once we have this, the smooth Livshits theorem applies and gives us a smooth function u_s with $F_s(u_s) = \beta_s$. Using the magnetic Carleman estimate above, it is clear that u_s must have degree one less than β_s, so we can write

$$u_s = u^s_{-1} + u^s_0 + u^s_1.$$

In fact, let

$$\eta^{\pm}_s := \frac{X_s \mp iX^{\perp}_s}{2}.$$

Using the equations for Cartan's moving frame, we have that η^+ raises degree while η^- lowers degree. It's not hard to see with this set up that for $\delta_s = u^s_{-1} + u^s_1$ we have

$$X_s(u_s) = \beta_s.$$

Finally, it is not hard to show that u_s is a 1-form using either differential equations or by direct computation, and this completes the sketch of the proof of the existence of a smooth family of isometries $\{f_s\}$ with $f^*_s(g_0) = g_s$.

The next challenge is to show that these isometries are the "right ones," in the sense that $f^*_s(b_0) = b_s$. To do this, we simply conjugate by the diffeomorphisms to fix the metric and consider the case where only the intensities are varying. We can derive the following "non-homgeneous Jacobi equation" for the variation along a given magnetic geodesic γ for the system (g_0, b_0):

$$\ddot{y} + K^{g,b}y = \frac{d}{ds}\Big|_{s=0} b_s, \quad \dot{x} = by,$$

where the variational field associated to the systems is given by $S(t) = x(t)\dot{\gamma}(t) + y(t)\dot{\gamma}^{\perp}(t)$. It's not hard to see that this can be generalized to a vector field on S^gM, so x and y can be considered as smooth functions on S^gM. Using the magnetic Carleman estimates, one can show that y and Fy has finite degree, and in particular we deduce that y is constant and $by \equiv 0$. Either $b \equiv 0$, which we conclude that $b_s \equiv 0$ identically as geodesics are the unique length minimizers in their free homotopy class, or $b(x) = 0$ for some $x \in M$ and thus $y = 0$. This lets us conclude the derivative at $s = 0$ is zero, but the choice of $s = 0$ was arbitrary.

The question now is whether or not this result can be generalized globally. So far, we can say the following. It seems very likely that it can be generalized, but this is an ongoing project with Jacopo de Simoi, Valerio Assenza, and Ivo Terek.