Markov Chains, Mixing Times, and Couplings

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Motivation

Motivating Question

Preforming a random walk on some graph structure, how long does it take until you are "sufficiently random?"

Theorem (Diaconis, Bayer '92)

If you riffle shuffle a deck of size n, it takes approximately $\frac{3}{2}\log_2(n)$ shuffles until the deck is "sufficiently random."

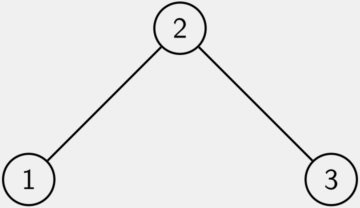
Graph Theory

Graph

We define a **graph** to be a tuple $G = (V, E)$ such that V is a collection of objects called **vertices** and $E \subseteq V \times V$ is a collection of pairs called **edges**.

Example

$V = \{1, 2, 3\}$, $E = \{(1, 2), (2, 3)\}$.



Graph Theory

Degree

We define the **degree** of a vertex to be the number of **neighbors**, or vertices which are connected by an edge, the vertex has. This is generally denoted by $\deg(x)$.

Regular

A graph is said to be n-**regular** if the degree of all the vertices is n.

Graph Theory

Example

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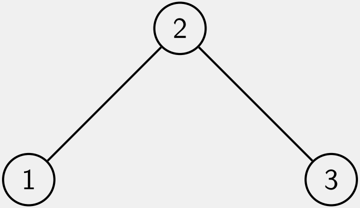
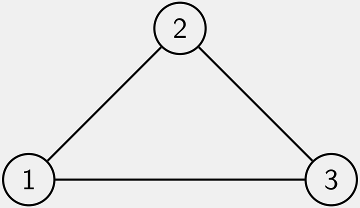


We see $\deg(2) = 2$, $\deg(1) = 1$, and $\deg(3) = 1$. This is therefore **not** regular.

Example

$V = \{1, 2, 3\}$, $E = \{(1, 2), (1, 3), (2, 3)\}$.



We see $\deg(2) = 2$, $\deg(1) = 2$, and $\deg(3) = 2$. This is therefore **2-regular**.

Markov Chains

Markov Property and Markov Chain

A **Markov Chain** is a series of random variables $(X_0, X_1, \ldots)$ on a common state space Ω satisfying the **Markov Property**:

$$\mathbf{P}\{X_n = x_n \mid X_1 = x_1, \ldots, X_{n-1} = x_{n-1}\} = \mathbf{P}\{X_n = x_n \mid X_{n-1} = x_{n-1}\}.$$

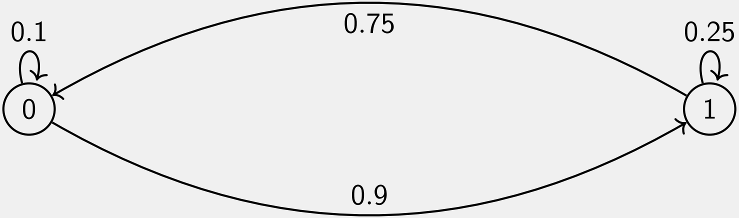
Transition Matrix

We can model Markov Chains using a **transition matrix**, which is a matrix with entries

$$P(x, y) = \mathbf{P}\{X_n = y \mid X_{n-1} = x\}$$

Markov Chains

Example Graph



Example Markov Chain

This Markov chain has transition matrix

$$P = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \left[\begin{array}{cc} 0.1 & 0.9 \\ 0.75 & 0.25 \end{array} \right] \end{array}$$

Aperiodic and Irreducible

We say our Markov Chain is **irreducible** if there exists a $t > 0$ for all $x, y \in \Omega$ such that

$$P^t(x, y) > 0.$$

We say that our Markov Chain is **aperiodic** if

$$\gcd\{t \geq 1 \mid P^t(x, x) > 0\} = 1$$

for all $x \in \Omega$.

Stationary Distribution

If our Markov chain is **irreducible**, then we have that there exists a unique distribution π such that

$$\pi P = \pi.$$

We call such a distribution a **stationary distribution**.

Limiting Distribution

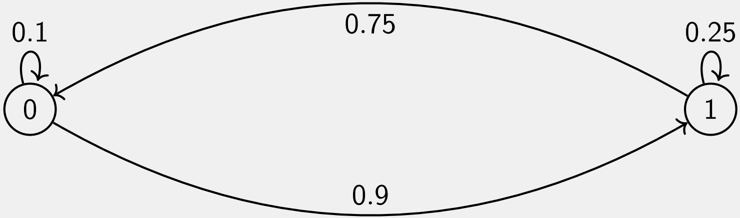
We call a distribution $\hat{\pi}$ a **limiting distribution** if

$$\lim_{t \to \infty} P^t(x, y) = \hat{\pi}(y).$$

If our distribution is **aperiodic** and **irreducible**, then we have that the stationary distribution π is the limiting distribution $\hat{\pi}$.

Markov Chains

Example Graph



Example Stationary Distribution

Notice that this is **aperiodic** and **irreducible**, and so we have a stationary distribution. The stationary distribution is

$$\pi = \begin{bmatrix} \frac{5}{11} & \frac{6}{11} \end{bmatrix}$$

Simple Random Walk

Given some graph G, we can define a **simple random walk on** G to be a Markov chain with state space V and transition matrix

$$P(x,y) = \begin{cases} \frac{1}{\deg(x)} & \text{if x and y are neighbors,} \\ 0 & \text{otherwise.} \end{cases}$$

Lazy Random Walk

Given some graph G, we can define a **lazy random walk on** G to be a Markov chain with state space V and transition matrix

$$P(x,y) = \begin{cases} \frac{1}{2} & \text{if } x = y, \\ \frac{1}{2\deg(x)} & \text{if x and y are neighbors,} \\ 0 \text{ otherwise.} \end{cases}$$

Mixing Times

Total Variation Distance

We define the **total variation distance** of two probability distributions μ and ν on a common state space Ω to be

$$||\mu - \nu||_{TV} = \max_{A \subseteq \Omega} |\mu(A) - \nu(A)|.$$

In particular, we care about

$$d(t) := ||P^t(x, \cdot) - \pi(\cdot)||_{TV}$$

Mixing Time

We define the **mixing time** of a Markov chain to be

$$t_{\text{mix}}(\epsilon) := \min\{t \mid d(t) \leq \epsilon\}.$$

Coupling

Markovian Coupling of Markov Chains

We define a **Markovian coupling of two Markov chains** (X_t) and (Y_t) with common state space Ω and transition matrix P to be the process $(X_t, Y_t)_{t=0}^{\infty}$ over $\Omega \times \Omega$, with the addendum that

$$\mathbf{P}\{X_{t+1} = x' \mid X_t = x, Y_t = y\} = P(x, x')$$

and

$$\mathbf{P}\{Y_{t+1} = y' \mid X_t = x, Y_t = y\} = P(y, y').$$

We will also require that $X_s = Y_s$ for some s implies $X_t = Y_t$ for all $t \geq s$. A coupling is not a required to be Markovian (and it may not even be the optimal coupling), but in general we want our Markov chains to be Markovian.

Coupling

Theorem

Let

$$\tau := \min\{t \mid X_s = Y_s \text{ for all } s \geq t\}.$$

Then we have

$$d(t) \leq \max_{x,y\in\Omega} ||P^t(x,\cdot) - P^t(y,\cdot)||_{TV}$$

$$\leq \max_{x,y\in\Omega} \mathbf{P}\{\tau > t \mid X_0 = x, Y_0 = y\}$$

$$\leq \max_{x,y\in\Omega} \frac{\mathbf{E}(\tau \mid X_0 = x, Y_0 = y)}{t}$$

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Coupling

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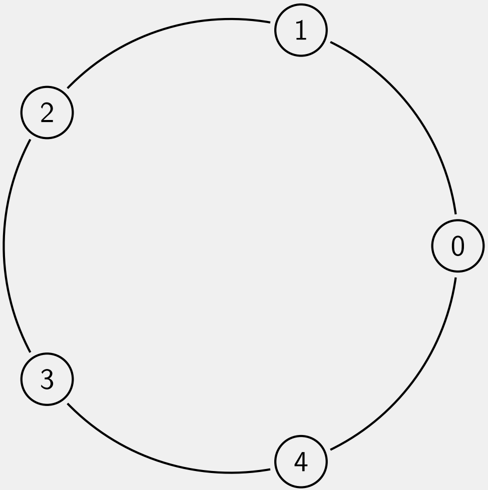

Coupling

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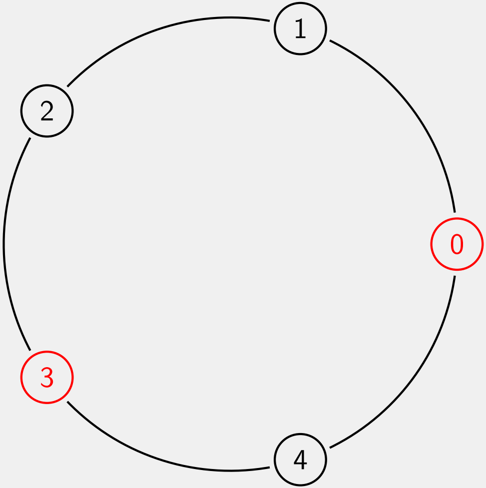

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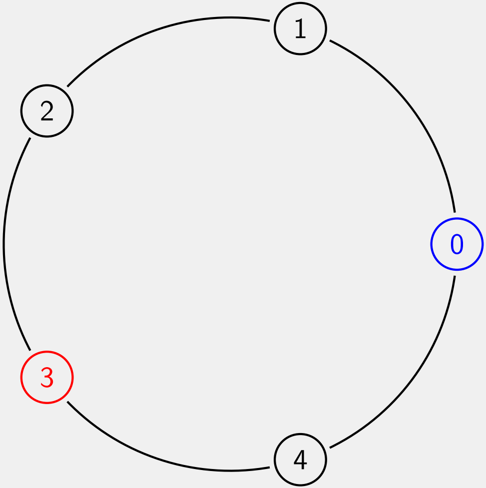

Coupling

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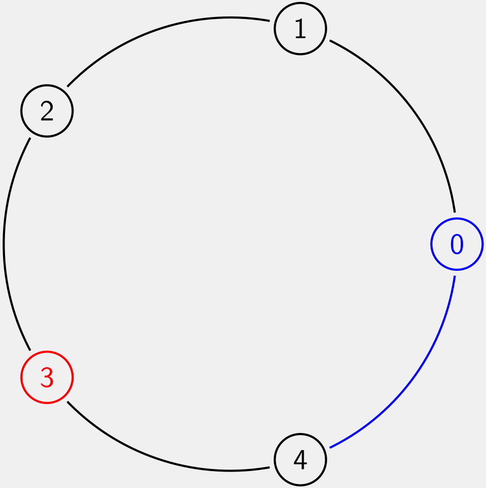

Coupling

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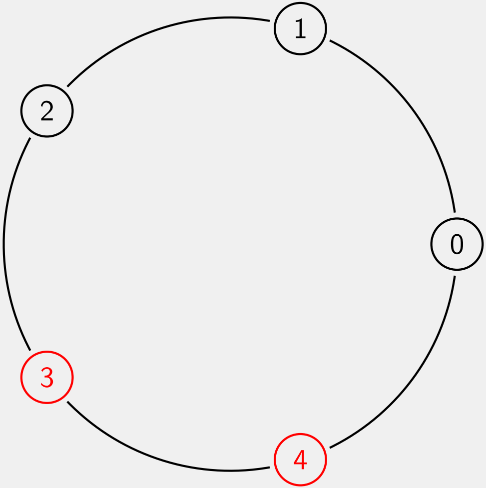
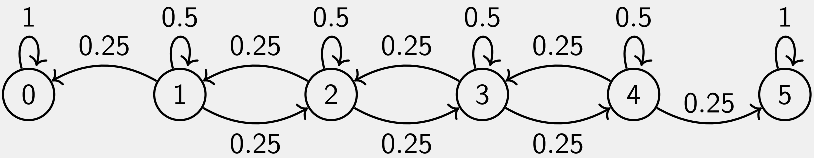


Coupling

Example

If we measure the **clockwise distance** between the two walkers, this coupling gives us a new Markov chain on $\{0, 1, \ldots, n\}$ to study.

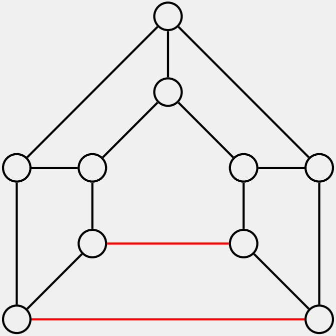


This gives us

$$d(t) \leq \frac{n^2}{4t} \to t_{\text{mix}}(\epsilon) \leq \frac{n^2}{4\epsilon}$$

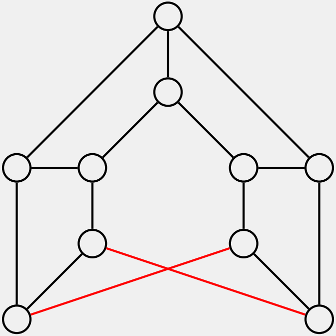
3-Regular Graphs

Prism and Möbius ladder graphs



3-Regular Graphs

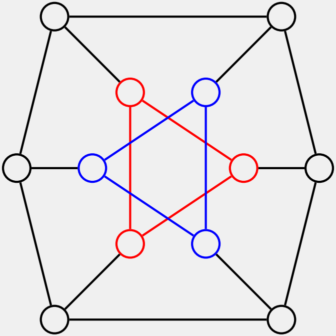
Prism and Möbius ladder graphs



3-Regular Graphs

GP(n,k)

Below is an example of $GP(6, 2)$.



Notice that if we imagined a cycle on the inside, the nodes which are the same color would be distance **2** away from eachother.

3-Regular Graphs

Results

Using coupling, we were able to determine that for Möbius ladder graphs and prism graphs of size n, the mixing time for a (slightly modified) lazy random walk is bounded by

$$t_{\text{mix}}(\epsilon) \leq \frac{3n^2}{16\epsilon} + \frac{6}{\epsilon},$$

and for the generalized Petersen graph $\text{GP}(n, k)$, the mixing time of the (slightly modified) lazy random walk is bounded by

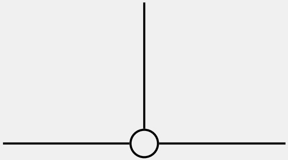
$$t_{\text{mix}}(\epsilon) \leq \frac{3|k|^2}{2\epsilon} + \frac{3}{2\epsilon}\left(\frac{n}{|k|}\right)^2 + \frac{15}{\epsilon},$$

where $|k| = n/\gcd(n, k)$.

3-Regular Graphs

"Triangulating"

By "triangulating" a 3-regular graph, we mean replace each vertex with a complete graph of size 3.



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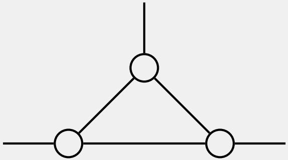

3-Regular Graphs

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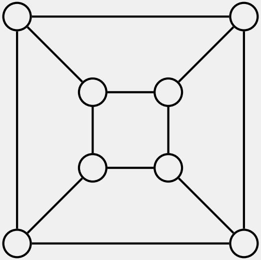
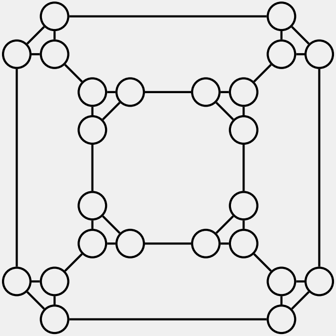


3-Regular Graphs

Example



3-Regular Graphs

Results

We found that when you triangulate the Möbius ladder graphs and prism graphs of size n, your mixing time transforms into

$$t_{\text{mix}}(\epsilon) \leq \frac{15n^2}{16\epsilon} + \frac{87}{5\epsilon}.$$

For the generalized Petersen graph $\text{GP}(n, k)$, it transforms into

$$t_{\text{mix}}(\epsilon) \leq \frac{15|k|^2}{2\epsilon} + \frac{15}{2\epsilon}\left(\frac{n}{|k|}\right)^2 + \frac{9}{\epsilon}\left(\frac{n}{|k|}\right) + \frac{9}{\epsilon}|k| + \frac{108}{\epsilon},$$

where $|k| = n/\gcd(n, k)$.

3-Regular Graphs

Remaining Questions

- Can we generalize this result to all vertex transitive 3-regular graphs?
- Can we extend it to all 3-regular graphs?
- Are the bounds we found above tight, or can we improve them?
- Does a similar result apply to lower bounds on these mixing times?

References

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Richard Durrett and R Durrett, *Essentials of stochastic processes*, vol. 1, Springer, 1999.

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Acknowledgements

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