A counterexample to a marked length spectrum semi-rigidity problem

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Introduction

Throughout, let M be a smooth closed manifold.

Fact

If g is a Riemannian metric on M with everywhere negative sectional curvature, then inside of every free homotopy class there exists a unique closed geodesic.

Definition

Let M be smooth closed manifold and let g be a Riemannian metric on M with everywhere negative sectional curvature. The *marked length spectrum* is a function MLS_g which takes a free homotopy class σ and returns the length of the unique closed geodesic γ_σ inside of it.

Conjecture (Burns-Katok [2])

If g and g' are two negatively curved metrics with $\text{MLS}_g = \text{MLS}_{g'}$, then g and g' are isometric.

While the conjecture is open in full generality, it is known to be true in the following cases:

- if M is a surface (Croke [4], Otal [9]);
- if the manifold is locally-symmetric (Besson-Courtois-Gallot [1], Hamenstädt [8]);
- if the metrics are close in a sufficiently fine topology (Guillarmou-Lefeuvre [10]).

Some "semi-rigidity" problems have been considered alongside of the original problem:

- Does $\text{MLS}_g \approx \text{MLS}_{g'}$ implies $g \approx g'$? (Butt [3])
- Does $\text{MLS}_g \leq \text{MLS}_{g'}$ imply $\text{Vol}(g) \leq \text{Vol}(g')$? (Croke-Dairbekov-Sharafutdinov [6])

The aim is to study a new kind of semi-rigidity problem.

Motivating Questions

Definitions

Let M be a smooth closed orientable surface, let g and g' be two metrics on M, and let $f : (M, g) \to (M, g')$ be a diffeomorphism.

1. We say that f is *volume shrinking* if $\text{Jac}(f) \leq 1$.
2. We say that f is *length shrinking* if $\|D_x f(v)\| \leq \|v\|'$ for all $(x, v) \in TM$.

Questions

1. Does $\text{MLS}_g \leq \text{MLS}_{g'}$ imply there is a volume shrinking diffeomorphism?
2. Does $\text{MLS}_g \leq \text{MLS}_{g'}$ imply there is a length shrinking diffeomorphism?

Volume Shrinking Case

It is easy to show that the answer to the first question is "yes" on a surface by combining two known results.

Theorems

1. (Croke-Dairbekov [5]) If M is a smooth closed orientable surface and g and g' are two negatively curved metrics on M with $\text{MLS}_g \leq \text{MLS}_{g'}$, then $\text{Vol}(g) \leq \text{Vol}(g')$.
2. (Moser [11]) If ω and ω' are two volume forms with $\text{Vol}(\omega) = \text{Vol}(\omega')$, then there is a diffeomorphism $f : M \to M$ so that $f^*(\omega') = \omega$.

Length Shrinking Case – Main Result

We show that the answer to the second question is "no" in a rather strong sense.

Theorem (Gogolev-Marshall Reber '23 [7])

Let M be a closed, connected, orientable surface. If g is a negatively curved metric on M, then there exist arbitrarily C^∞-small perturbations g' of g for which there exists an $\epsilon > 0$ so that

- $\text{MLS}_{g'} > (1 + \epsilon)\text{MLS}_g$, and
- there does not exist a length shrinking diffeomorphism $f : M \to M$.

Preliminaries

Throughout,

- let M be as above and fix a negatively curved metric g,
- let $\mathcal{F}$ be the collection of shortest g-geodesics with a self-intersection; we'll refer to these as "figure eights,"
- for each $\gamma \in \mathcal{F}$, let γ^1 denote the shorter loop of the figure eight and γ^2 the longer loop of the figure eight,
- let $\gamma_{\text{Short}} \in \mathcal{F}$ be such that $\ell_g(\gamma^1_{\text{Short}}) \leq \ell_g(\gamma^1)$ for every $\gamma \in \mathcal{F}$, where ℓ_g denotes the length of a curve with respect to the metric g.

The following three results are established in [7].

Claim 1

There exists a C^∞ neighborhood U of g such that for every $g' \in U$, every length shrinking diffeomorphism $f : M \to M$, and every $\gamma \in \mathcal{F}$, we can find an $\eta \in \mathcal{F}$ so that $f \circ \gamma$ is homotopic to η.

- This allows us to choose metrics so that $\gamma_f := f \circ \gamma$ is in the same free homotopy class as a figure eight.

Claim 2

One can perturb g' in such a way so that

- each $\gamma \in \mathcal{F}$ is a g'-geodesic after reparamterization,
- there are constants $0 < \xi_1 < \xi_2$ so that for every $\gamma \in \mathcal{F}$ we have
$$\ell_{g'}(\gamma^1) = \ell_g(\gamma^1) - \xi_1 \text{ and } \ell_{g'}(\gamma^2) = \ell_g(\gamma^2) + \xi_2,$$
- there is an $\epsilon > 0$ so that $\text{MLS}_{g'} > (1 + \epsilon)\text{MLS}_g$.

The perturbation we construct allows us to adjust the constants ξ_1 and ξ_2 so that the above properties hold for all ξ_2 close to ξ_1.

- This allows us to perturb g to get new metrics g' so that they uniformly shrink one loop of the figure eights and uniformly expand the other loop of the figure eights. Moreover, this can be done in such a way so that the marked length spectrum is getting bigger.

Claim 3

For small enough perturbations according to the previous claims, we have that for *any* shrinking diffeomorphism the intersection points are mapped close to one another.
More precisely, g' can be chosen so that for any shrinking diffeomorphism $f : M \to M$ and any $\gamma \in \mathcal{F}$, if $\eta \in \mathcal{F}$ is homotopic to $\gamma_f := f \circ \gamma$, then letting p be the intersection point of η and q the intersection point of γ_f, we have $d_g(p, q) < \xi_1/2$.

- This allows for us to say that every shrinking diffeomorphism maps a figure eight uniformly close to another figure eight.

Sketch of the Proof

Assuming the preliminary claims, we are able to sketch a proof of the main result.

Proof

Let g' be close to g according to Claims 1 through 3. Suppose for contradiction there is a shrinking diffeomorphism $f : M \to M$ between g and g'. Let γ_f and η be as in Claim 3, and suppose without loss of generality that γ_f^1 is homotopic to η^1.

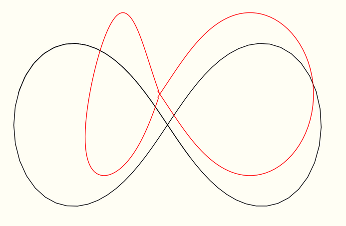

Figure 1. An example of γ_f in red and η in black.

Let ν be the unique g-geodesic connecting p and q. By Claim 3, we have $\ell_g(\nu) < \xi_1/2$. Concatenating γ_f^1 with ν, ν^{-1}, and η^2, we get a new figure eight curve in the same free homotopy class as η. Using the length shrinking property, the first loop has length
$$\ell_g(\nu^{-1}\gamma_f^1\nu) < \xi_1 + \ell_g(\gamma_f^1) \leq \xi_1 + \ell_{g'}(\gamma^1_{\text{Short}}) = \ell_g(\gamma^1_{\text{Short}}) \leq \ell_g(\eta^1).$$
This contradicts the fact that η is a length minimizer in its free homotopy class.

References

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