A POSITIVE PROPORTION LIVSHITS THEOREM – MARYLAND

1. Notes for Talk

Joint work with Caleb Dilsavor

Throughout this talk,

- let M be a closed connected Riemannian manifold, and
- let $f^t : M \to M$ be a transitive Anosov diffeomorphism or flow.

Recall the following.

- A Hölder continuous map $\varphi : M \to \mathbb{R}$ is a *coboundary* if there exists a Hölder continuous $\kappa : M \to \mathbb{R}$ such that
$$\varphi = \begin{cases} \kappa \circ f - \kappa & \text{if } f \text{ is a diffeomorphism,} \\ \left.\dfrac{d}{dt}\right|_{t=0} (\kappa \circ f^t) & \text{if } f^t \text{ is a flow.} \end{cases}$$
- Let P be the collection of all closed orbits for the system f^t. We denote the *period* of a closed orbit by $\ell(\gamma)$.
- Given a Hölder continuous function φ, we define the *φ-period* of an orbit $\gamma \in P$ by
$$\ell_\varphi(\gamma) := \begin{cases} \displaystyle\sum_{k=0}^{\ell(\gamma)-1} \varphi(f^k(x_\gamma)) & \text{if } f \text{ is a diffeomorphism,} \\ \displaystyle\int_0^{\ell(\gamma)} \varphi(f^s(x_\gamma))\, ds & \text{if } f^t \text{ is a flow.} \end{cases}$$

It is clear that if φ is a coboundary, then $\ell_\varphi \equiv 0$. The celebrated Livshits theorem, proven by Alexander Sasha Livshits in the 70's, tells us that the converse is also true – namely, if $\ell_\varphi \equiv 0$, then φ is a coboundary. The goal of today's talk is to discuss a generalization of this theorem in certain scenarios. To make things easy, we'll focus on the case where f is a transitive Anosov diffeomorphism, and we'll mention towards the end how this result generalizes to the flow case. A consequence of our main result says that one only needs to verify that $\ell_\varphi \equiv 0$ on a set of positive asymptotic upper density:

$$\limsup_{n\to\infty} \frac{|\{\gamma \in P \mid \ell_\varphi(\gamma) = 0 \text{ and } \ell(\gamma) = n\}|}{|\{\gamma \in P \mid \ell(\gamma) = n\}|} > 0 \implies \varphi \text{ is a coboundary.} \tag{1}$$

The big idea behind this proof is the "orbital central limit theorem" (abbreviated CLT). Let $P(n)$ be the set of closed orbits with period n.

- We define the *Bowen measures* on the set $P(n)$ by
$$\mu_n := \frac{\sum_{\gamma \in P(n)} \delta_\gamma}{|P(n)|},$$
where δ_γ is the Dirac measure along the orbit γ.

- Given a Hölder continuous function φ, we define the *dynamical variance* of φ by

$$\sigma_\varphi^2 \coloneqq \lim_{n\to\infty} \frac{1}{n}\mu\left((S_n(\varphi) - \mu(S_n(\varphi)))^2\right), \text{ where } S_n(\varphi)(x) \coloneqq \sum_{k=0}^{n-1} \varphi(f^k(x)).$$

The CLT says that

$$\mu_n\left(a \le \frac{\ell_\varphi - n\mu(\varphi)}{\sqrt{n}} < b\right) \xrightarrow{n\to\infty} \frac{1}{\sqrt{2\pi}\sigma_\varphi}\int_a^b e^{-t^2/2\sigma_\varphi^2}dt,$$

provided $\sigma_\varphi^2 > 0$. Using this, along with a classic result by Ratner which says that $\sigma_\varphi^2 = 0$ if and only if there is a constant $C \in \mathbb{R}$ such that $\varphi - C$ is a coboundary, one can prove Equation (1) by showing that

(1) one gets a contradiction using the CLT if one assumes that ℓ_φ is zero on a set of positive proportion and $\sigma_\varphi^2 > 0$,
(2) since $\sigma_\varphi^2 = 0$, we have $\varphi - C = \kappa \circ f - \kappa$, and summing this over an orbit on which φ vanishes gives us $C = 0$.

We remark that the CLT as we've stated it dates back to an exercise in Ruelle's book on thermodynamic formalism (1978). A weighted version of this was proved by Coelho and Parry in 1990. For flows, the story behind the CLT is a little different. Now let f^t be a transitive Anosov flow. Given $\Delta > 0$, let $P(T, \Delta)$ be the collection of orbits whose period lies in $(T, T + \Delta]$.

- We define the *Bowen measures* on $P(T, \Delta)$ by

$$\mu_{T,\Delta} \coloneqq \frac{\sum_{\gamma\in P(T,\Delta)} \delta_\gamma}{|P(T, \Delta)|}.$$

- Define the *dynamical variance* in the same way, replacing n with T and $S_n(\varphi)$ with

$$S_T(\varphi)(x) \coloneqq \int_0^T \varphi(f^s(x))ds.$$

It is unknown whether the CLT holds for flows in general. It was shown by Cantrell and Sharp in 2021 that the CLT holds for Anosov flows whose stable and unstable distributions are not jointly integrable. Dilsavor and I proved a weighted version of this our recent preprint with the same title. As long as the CLT holds, we can use the above argument to prove the positive proportion Livshits theorem.

I'll finish the talk by discussing two interesting applications of the positive proportion Livshits theorem.

(1) The first is related to marked length spectrum rigidity. Recall that if (M, g) is a negatively curved manifold, then in every free homotopy class there exists a unique closed geodesic of minimal length. The *marked length spectrum* of the metric is the function MLS_g on the space of free homotopy classes which returns the length of the closed geodesic. It was conjectured by Burns and Katok that the marked length spectrum determines a metric up to isometry. In 1990, this was proven to be true in the case where M is a surface by Otal and Croke (separately). In 2020, Sawyer generalized the surface case by showing that one only needs to check that the marked length spectrum of two metrics agrees on a set of free homotopy classes whose complement grows subexponentially with respect to length. Using the positive proportion Livshits theorem for flows, we are able to improve this result and show

that one only has to check that two metrics agree on a set of free homotopy classes with positive proportion.

(2) The second is related to a recent rigidity result by Gogolev and Rodriguez Hertz, and is actually the primary motivation for the result. In their recent preprint, they showed that if two smooth transitive Anosov flows on a 3-dimensional manifold are C^0 conjugate, and at least one of them is not a constant roof suspension, then they are actually C^∞ conjugate. One ingredient in their proof is a weighted version of the positive proportion Livshits theorem.

2. Questions

(1) (D. Dolgopyat)
Question: Can you find an example of a Hölder continuous potential which vanishes on a dense set of periodic orbits? Moreover, can you find examples with different regularities?
Answer: Yes, take the full shift on two symbols and define $\varphi : \Sigma \to \mathbb{R}$ with $\varphi(x) = (-1)^{x_0}$. This vanishes on a dense set of orbits, however it is not a coboundary by the usual Livshits theorem. For more examples, we can try looking at Section 7.4 in here.

(2) (J. DeWitt) For simplicity, let $f : M \to M$ be a transitive Anosov diffeomorphism. Let P be the periodic orbits, and let $Q \subseteq P$ be some collection. To every periodic orbit we can associated an invariant probability measure, $\gamma \mapsto \widehat{\delta}_\gamma$. Let $\widehat{Q}$ denote the set of these measures and let $\mathcal{M}_1(f)$ be the set of invariant probability measures. If $\widehat{Q} \subseteq \mathcal{M}_1(f)$ is dense in the weak* topology, then we know that if φ is a Hölder potential such that $\widehat{Q} \cdot \varphi = \{0\}$, then φ must be a coboundary (this is the usual Livshits theorem).
Question: Suppose we only know that $\widehat{Q} \subsetneq \mathcal{M}_1(f)$ (i.e. the sets may not be equal). Can we find a Hölder continuous function φ so that $\widehat{Q} \cdot \varphi = \{0\}$ but φ is not a coboundary?
Answer: Let $\widehat{P}(n)$ be the collection of periodic orbits with period equal to n and let $\widehat{Q}(n) := \widehat{Q} \cap \widehat{P}(n)$. It's clear that the central limit theorem should give us that if $\widehat{Q}(n)$ has positive proportion, then $\widehat{Q} \cdot \varphi = \{0\}$ implies φ is a coboundary. We will use the central limit theorem to also construct an example of a set $\widehat{Q}$ which is not weak* dense but which determines whether a function is a coboundary.

Let μ be the MME, $\varphi : M \to \mathbb{R}$ be Hölder with $\mu(\varphi) = 0$ and $\sigma_\varphi^2 > 0$. Let μ_n be the sequence of Bowen measures. For $\epsilon > 0$, let

$$U_\epsilon := \{\gamma \in P \mid |\delta_\gamma(\varphi)| < \epsilon\},$$

where δ_γ is the probability measure on the periodic orbit γ. Notice this is the periodic measures which are in a small neighborhood of the MME determined by φ. For $n \geq 1$, let

$$U_\epsilon^{\sqrt{n}} := \{\gamma \in P \mid |\delta_\gamma(\varphi)| < \sqrt{n}\epsilon\}.$$

Clearly $U_\epsilon \subseteq U_\epsilon^{\sqrt{n}}$ for each $n \geq 1$, and furthermore

$$\mu_n(U_\epsilon) \leq \mu_n(U_\epsilon^{\sqrt{n}}) = \mu_n(\ell_\varphi/\sqrt{n} \in (-\epsilon, \epsilon)).$$

By the central limit theorem,

$$\limsup_{n\to\infty} \mu_n(U_\epsilon) \leq \frac{1}{\sqrt{2\pi}\sigma_\varphi} \int_{-\epsilon}^{\epsilon} e^{-t^2/(2\sigma_\varphi^2)} dt.$$

Let Q_ϵ be the complement, so

$$\limsup_{n\to\infty} \mu_n(Q_\epsilon) \geq \frac{2}{\sqrt{2\pi}\sigma_\varphi} \int_{-\infty}^{-\epsilon} e^{-t^2/(2\sigma_\varphi^2)} dt.$$

For any $\epsilon > 0$ this is positive, so in particular this set will have positive proportion. Moreover, the associated set of measures has positive proportion, and by construction this set cannot be weak* dense (as it avoids an open neighborhood of μ).

(3) (J. Marshall Reber) For simplicity, let $f : M \to M$ be a transitive Anosov diffeomorphism. It is natural to consider cocycles generated by functions of the form $\varphi : M \to \mathrm{GL}(d, \mathbb{R})$; for example of why this is natural to consider, see here.
Question: Can one extend our results to cocycles of this form?
One immediate obstacle is a generalization of Ratner's theorem...