A POSITIVE PROPORTION LIVSHITS THEOREM – NORTHWESTERN

Joint work with Caleb Dilsavor

0.1. **Introduction and review.** Throughout this talk, let M be a closed connected Riemannian manifold and let $f^t : M \to M$ be a transitive Anosov diffeomorphism or flow.

- Recall that a Hölder continuous map $\varphi : M \to \mathbb{R}$ is a *coboundary* if there exists a Hölder continuous $\kappa : M \to \mathbb{R}$ such that

$$\varphi = \begin{cases} \kappa \circ f - \kappa & \text{if } f \text{ is a diffeomorphism,} \\ \left.\dfrac{d}{dt}\right|_{t=0} (\kappa \circ f^t) & \text{if } f^t \text{ is a flow.} \end{cases}$$

- Let P be the collection of all closed orbits for the system f^t. We denote the (dynamical) length of a closed orbit by $\ell(\gamma)$ (this is the minimal period of an orbit).
- We define the *period* of the orbit γ with respect to the function φ by

$$\ell_\varphi(\gamma) := \begin{cases} \displaystyle\sum_{k=0}^{\ell(\gamma)-1} \varphi(f^k(x_\gamma)) & \text{if } f \text{ is a diffeomorphism,} \\ \displaystyle\int_0^{\ell(\gamma)} \varphi(f^s(x_\gamma))\, ds & \text{if } f^t \text{ is a flow.} \end{cases}$$

It is clear that if φ is a coboundary, then $\ell_\varphi \equiv 0$. The celebrated Livshits theorem, proven by Alexander Sasha Livshits in the 70's, tells us that the converse is also true.

Theorem 0.1 (Livshits, 1972)**.** *A Hölder continuous function φ is a coboundary if and only if $\ell_\varphi \equiv 0$.*

This theorem has a rich history in rigidity theory within dynamical systems. For example:

- (Livshits–Sinai) A transitive Anosov system has an invariant volume measure if and only if the log Jacobian is a coboundary.
- If two transitive Anosov flows are orbit equivalent and the corresponding periodic orbits have the same period, then they are C^0-conjugate.

The goal of the talk today is to discuss a generalization of the Livshits theorem in certain cases. The organization of this talk is as follows:

(i) We will start by focusing on the case where $f : M \to M$ is a transitive Anosov diffeomorphism. We'll prove a baby case of our main result, which says that one only needs to verify the condition that $\ell_\varphi \equiv 0$ on a set of positive asymptotic upper density:

$$\limsup_{n\to\infty} \frac{|\{\gamma \in P \mid \ell_\varphi(\gamma) = 0 \text{ and } \ell(\gamma) = n\}|}{|\{\gamma \in P \mid \ell(\gamma) = n\}|} > 0 \implies \varphi \sim 0. \tag{1}$$

(ii) Next, we'll go into some detail on how we can extend this result to a weighted case, and we'll explain what weighted means in this context. In particular, the weighted flow version was necessary for the recent rigidity result by Gogolev and Rodriguez Hertz, which we'll discuss more towards the end.

(iii) After discussing the weighted diffeomorphism case, we'll start to go into detail on how the flow case works. Here, we have to start breaking things up into different cases and generalize some number theoretic techniques to the dynamical setting.
(iv) Finally, we'll end the talk by discussing applications of this result. If there's time, we'll also discuss some questions from others, as well as discuss a version of the non-positive Livshits theorem that we proved (and some of the challenges that came with it).

0.2. **The diffeomorphism case.**

0.2.1. *Unweighted.* The big idea behind this proof is the so-called *orbital central limit theorem* (abbreviated CLT throughout). For transitive Anosov diffeomorphisms, the unweighted CLT dates back to an exercise in Ruelle's textbook on thermodynamic formalism. Letting $f : M \to M$ be a transitive Anosov diffeomorphism and letting $P(n)$ be the collection of closed orbits with period n, we define the *(unweighted) Bowen measures* on $P(n)$ by

$$\mu_n := \frac{\sum_{\gamma \in P(n)} \delta_\gamma}{|P(n)|}.$$

Bowen's equidistribution result says that μ_n converges to the measure of maximal entropy in the weak* topology. The central limit theorem can be interpreted as giving more information about this convergence. If μ is the measure of maximal entropy, namely if μ is the measure which achieves the supremum in the variational principle, then we define the *dynamical variance* of φ with respect to μ by

$$\sigma_\varphi^2 := \lim_{n\to\infty} \frac{1}{n}\mu\left((S_n(\varphi) - \mu(S_n(\varphi))^2\right), \text{ where } S_n(\varphi) := \sum_{k=0}^{n-1} \varphi(f^k(x)).$$

If we consider $\ell_\varphi/\sqrt{n}$ as a random variable on $P(n)$, then the CLT says that if $\sigma_\varphi^2 > 0$ then this random variable converges in distribution to a normal random variable with mean zero and variance σ_φ^2, in the sense that

$$\lim_{n\to\infty} \mu_n\left(\frac{\ell_\varphi}{\sqrt{n}} \in (a,b)\right) = \frac{1}{\sqrt{2\pi}\sigma_\varphi}\int_a^b e^{-t^2/(2\sigma_\varphi^2)}dt.$$

Remark 0.2. We can compare this to Ratner's famous central limit theorem, which says

$$\lim_{n\to\infty} \mu\left(\frac{S_n(\varphi)}{\sqrt{n}} \in (a,b)\right) = P(N(0,\sigma_\varphi^2) \in (a,b)).$$

If one uses Bowen's equidistribution, one can write this as

$$\lim_{n\to\infty}\lim_{m\to\infty} \mu_m\left(\frac{S_n(\varphi)}{\sqrt{n}} \in (a,b)\right) = P(N(0,\sigma_\varphi^2) \in (a,b)).$$

Note that the CLT we're using comes from setting $n = m$ in the above, however this is not easy to justify.

The last ingredient is a classic result by Ratner, which states that $\sigma_\varphi = 0$ if and only if there is a constant $C \in \mathbb{R}$ so that $\varphi - C$ is a coboundary. Let's now assume that $\mu(\varphi) = 0$ (the other case is similar).

Step 1: Assume the positive proportion assumption and assume $\sigma_\varphi^2 > 0$. Let

$$0 < D := \limsup_{n\to\infty} \frac{|\{\gamma \in P \mid \ell_\varphi(\gamma) = 0 \text{ and } \ell(\gamma) = n\}|}{|\{\gamma \in P \mid \ell(\gamma) = n\}|} = \limsup_{n\to\infty} \mu_n\left(\frac{\ell_\varphi}{\sqrt{n}} = 0\right).$$

For $\epsilon > 0$, we have by the CLT and the above

$$D \leq \limsup_{n\to\infty} \mu_n\left(\frac{\ell_\varphi}{\sqrt{n}} \in (-\epsilon, \epsilon)\right) = \frac{1}{\sqrt{2\pi}\sigma_\varphi}\int_{-\epsilon}^{\epsilon} e^{-t^2/(2\sigma_\varphi^2)}dt.$$

The rightmost term goes to zero as $\epsilon \to 0^+$, and this gives us the contradiction. Thus we must have $\sigma_\varphi^2 = 0$.

Step 2: By Ratner, there is a $\kappa : M \to \mathbb{R}$ Hölder continuous so that $\varphi = C + \kappa \circ f - \kappa$. Since $D > 0$ in the above, we can find an orbit $\gamma \in P$ such that $\ell_\varphi(\gamma) = 0$. Notice

$$0 = \ell_\varphi(\gamma) = \sum_{k=0}^{\ell(\gamma)-1} \varphi \circ f(x) = \ell(\gamma)C \text{ for some } x \in \gamma.$$

We conclude that $C = 0$, so φ is a coboundary.

0.2.2. *Weighted.* The story is mostly the same for the weighted case, one just has to understand what "weighted" means. Let $\psi : M \to \mathbb{R}$ be Hölder continuous. We define the *weighted Bowen measures* by

$$\mu_{n,\psi} := \frac{\sum_{\gamma\in P(n)} \exp(\ell_\psi(\gamma))\delta_\gamma}{\sum_{\gamma\in P(n)} \exp(\ell_\psi(\gamma))}.$$

We recover the usual unweighted case by just considering $\psi \equiv 0$. In this case, Bowen's equidistribution result says that $\mu_{n,\psi} \xrightarrow{n\to\infty} \mu_\psi$, where μ_ψ is the equilibrium states associated to ψ and the convergence is in the weak* topology. One then replaces the measure of maximal entropy with the equlibrium state in all of the prior definitions, and, using the weighted CLT proved by Coelho and Parry in 1990, the same argument proves the result in this case as well.

0.3. **The flow case.** The flow case becomes significantly harder. One main issue comes from understanding the transfer operator in the flow case. Observe that if the flow is just a constant roof suspension, then we can use the prior diffeomorphism case to prove the positive proportion Livshits theorem here. Thus, we will assume our flows are not just constant roof suspensions.

Many recent developments in the flow case come from noticing that there is an analogy between the distribution of primes and the distribution of closed orbits, thus allowing us to abuse analytic number theory techniques to get results on the distribution of closed orbits of transitive Anosov flows. To bridge these two concepts, however, one has to assume that a certain approximability condition holds for the flows. In the case of transitive Anosov flows, if one assumes that the stable and unstable distributions are *not* jointly integrable, then one can use techniques pioneered by Dolgopyat to push these results from the number theory setting to the dynamical setting. The real power of this technique can be seen in a 1999 paper by Pollicott and Sharp ("Error terms for closed orbits of hyperbolic flows"), in which a classic result by Margulis on the growth rate of closed orbits was rederived and generalized with number theoretic techniques. Many of our techniques are based on the tools introduced in this paper.

More precisely, Cantrell and Sharp showed in 2018 that there is an unweighted CLT in this setting. Fixing $\Delta > 0$ and letting $P(T, \Delta)$ be the set of closed orbits whose length lies in $(T, T+\Delta]$, we consider probability measures on $P(T,\Delta)$ given by

$$\mu_{T,\Delta} := \frac{\sum_{P(T,\Delta)} \delta_\gamma}{|P(T,\Delta)|}.$$

The dynamical variance has a natural analogue in this setting:

$$\sigma_\varphi^2 := \lim_{T\to\infty} \frac{1}{T}\mu\left((S_T(\varphi) - \mu(S_T(\varphi))^2\right), \text{ where } S_T(\varphi) := \int_0^T \varphi(f^s(x))ds.$$

We now assume that the unstable and stable distributions of the flow are not jointly integrable, $\mu(\varphi) = 0$, and $\sigma_\varphi^2 > 0$. The central limit theorem now says that if we consider $\ell_\varphi/\sqrt{T}$ as a random variable on $P(T,\Delta)$, then the random variable converges in distribution to a normal distribution with mean zero and variance σ_φ^2. Ratner's result still holds here, and so the flow version of Equation (1) holds.

The Plante conjecture, stated in 1972, states that if the stable and unstable distributions are not jointly integrable, then the flow is a constant roof suspension. This conjecture was verified by Plante in the setting of 3-dimensional volume preserving flows, but (as far as I know) nothing is known in higher dimensions.

Finally, I'll mention that we also generalized the Cantrell Sharp CLT to the weighted scenario. All definitions are similar to those as in the weighted diffeomorphism. In proving the weighted CLT, we had to generalize the results of Pollicott and Sharp to the weighted scenario as well, and thus we also showed a weighted prime orbit theorem, generalizing Margulis even further.

Before moving on, I'd also like to remark that, while I've stated everything for Anosov systems, all of the results hold for symbolic systems and axiom A systems. Much of the arguments actually come from the symbolic setting using Markov partitions, but for aesthetic purposes we'll not get into the details of this.

0.4. **Applications.** Beyond just generalizing the Livshits theorem, there are two main applications of the result. I'll discuss those now.

Application 1: One application of the classical Livshits theorem is in marked length spectrum rigidity. Recall that on a negatively curved manifold there exists a unique closed geodesic in every non-trivial free homotopy class. The *marked length spectrum* is a function on the space of free homotopy classes which returns the length of the closed geodesic. It was conjectured by Burns and Katok in 1985 that the marked length spectrum determines a metric up to isometry, and this was proven to be true for surfaces in 1990 by Otal and Croke separately. We briefly mention the steps of Otal's proof.

- Step 1: For a surface, the space of negatively curved metrics is path connected. Using Anosov structural stability, this implies that we always have an orbit equivalence between two negatively curved geodesic flows. Furthermore, one can choose this orbit equivalence so that it is homotopic to the identity. The marked length spectrum assumption then says that corresponding periodic orbits have the same length, so by our earlier note, we see that two metrics share the same marked length spectrum if they are C^0-conjugate. A result by Feldman and Ornstein lets us upgrade this result so that it is a smooth conjugacy.

Step 2: With this smooth conjugacy, we now need to construct an isometry. We do that by studying how the conjugacy behaves on fibers. A (now standard) argument by Otal shows us that h must come from a point map, and a few skips and a hop gives us that this point map is an isometry.

A result by Sawyer in 2020 showed that one can only needs to check that the marked length spectra of two metrics are equal on a set of free homotopy classes whose complement grows subexponentially with respect to length; she referred to his as *partial marked length spectrum rigidity*. The way this was done was by a careful analysis of Sigmund's proof of density of orbital measures in order to show that the orbit equivalence must be actually be a conjugacy, which is just an adjustment of step 1.

With the positive proportion Livshits theorem, we can now improve this and say that one only needs to check that the marked length spectra of two metrics are equal on a set of free homotopy classes with positive proportion.

Theorem 0.3 (Dilsavor and Marshall Reber, 2023). *Let M be a closed surface and let g_1, g_2 be two negatively curved metrics on M such that $\mathrm{MLS}_{g_1} = \mathrm{MLS}_{g_2}$ on a set of free homotopy classes with positive proportion. Then g_1 and g_2 are isometric.*

To exaggerate a little, this says that if the two metrics agree on even just 1% of free homotopy classes, then they must actually be isometric.

Application 2: Another application, and actually the primary motivation for this result, comes from the recent rigidity result by Gogolev and Rodriguez Hertz mentioned earlier, which uses the positive proportion Livshits theorem as an ingredient. A baby case of their result can be stated as follows.

Theorem 0.4 (Gogolev and Rodriguez Hertz, 2022). *Let $\dim(M) = 3$ and let $f_1^t, f_2^t : M \to M$ be smooth Anosov volume-preserving flows which are C^0-conjugate. Either*

(a) the conjugacy is smooth, or
(b) the flows are constant roof suspensions of Anosov diffeomorphisms.

One can view this as a generalization of Feldman and Ornstein to all (nice) Anosov flows, not just geodesic flows. One application of this result is a positive answer to a conjecture by Khalil and Lafont.

Theorem 0.5 (Gogolev and Rodriguez Hertz, 2022). *Let M be a closed surface, let g_1, g_2 be two negatively curved metrics on M. Let $h : S_{g_1}M \to S_{g_2}M$ be the orbit equivalence described earlier between the geodesic flows. Let $\varphi_1, \varphi_2 : M \to \mathbb{R}$ be smooth functions such that for every closed orbit γ_1 of g_1 we have*

$$\int_\gamma \varphi_1 = \int_{h_*(\gamma)} \varphi_2,$$

where h_* is the induced map on closed geodesics. Then there exists a $c > 0$ so that g_2 is isometric to $c^2 g_1$ via f and $\varphi_2 \circ f = c\varphi_1$.

Another application of the generalized Feldman-Ornstein result is to marked length spectrum rigidity in other contexts.

Theorem 0.6 (Marshall Reber, 2022)**.** *Let M be a closed surface and let (g_s, b_s) be a smooth family of magnetic systems such that the marked length spectrum is constant along s. Then there exists a smooth family of diffeomorphisms $f_s : M \to M$ such that $f_s^*(g_s) = g_0$ and $b_s \circ f_s = b_0$.*

I'll also mention that I expect the global version of this result to hold true; this is current work with myself, Valerio Delfino, Jacopo de Simoi, and Ivo Terek.

0.5. **Non-positive livshits theorem.**

Remark 0.7. Only get into this if there's time, since it's disjoint from our other stuff.

One can ask what happens when you replace the equality in the Livshits theorem with an inequality. Along the same lines, one can think about what happens when we have $\varphi \leq \kappa \circ f - \kappa$ for some $\kappa : M \to \mathbb{R}$; we call φ a *sub-coboundary* in this case. Notice that for every $\gamma \in P$, we get $\ell_\varphi(\gamma) \leq 0$. Even more remarkably, this also goes the other direction.

Theorem 0.8 (Lopes and Thieullen, 2003 and 2005)**.** *A Hölder continuous function is a sub-coboundary if and only if $\ell_\varphi \leq 0$.*

This result is useful for studying how the volume behaves under assumptions of the marked length spectrum. Namely, using the non-positive Livshits theorem, one can prove the following.

Theorem 0.9 (Croke and Dairbekov, 2002)**.** *Let M be a closed surface and let g_1, g_2 be two negatively curved metrics on M such that $\mathrm{MLS}_{g_1} \leq \mathrm{MLS}_{g_2}$. Then Vol$(g_1) \leq$ Vol(g_2), with equality if and only if g_1 is isometric to g_2.*

Somewhat surprisingly, the positive proportion assumption is not enough for the non-positive Livshits theorem. It is a good exercise to come up with a function $\varphi : M \to \mathbb{R}$ such that $\ell_\varphi \leq 0$ on a set of positive proportion, but φ is not a sub-coboundary. In fact, if you choose your example wisely, you can do this in such a way so that it has arbitrarily large positive weighted proportion. For a hint, you can see our preprint.

What we are able to show is that you only need to check the inequality on a collection of orbits whose complement has subexponential growth. The methods of proof for this are quite different and really get more into ergodic optimization than the central limit theorem. However, we are able to recover Noelle's result on partial marked length spectrum rigidity using this alternate method of proving it.

Theorem 0.10 (Dilsavor and Marshall Reber, 2023)**.** *Let M be a closed surface and let g_1, g_2 be two negatively curved metrics on M such that $\mathrm{MLS}_{g_1} \leq \mathrm{MLS}_{g_2}$ on a set of free homotopy classes whose complement has subexponential growth. Then Vol$(g_1) \leq$ Vol(g_2), with equality if and only if g_1 and g_2 are isometric.*

0.6. **Remaining Questions.** I'll list here some remaining questions. Attached are the people who either suggested the question or have given me some partial answers to the question (don't actually list their names during the presentation).

(1) (S. Cantrell and C. Dilsavor) Is this result optimal, in the sense that positive proportion is the best we can expect in general? Moreover, is the central limit theorem the right way to think about this?
(2) (D. Dolgopyat, A. Gogolev, S. Pavez) What are some counterexamples to a "dense" Livshits theorem?

(3) (J. DeWitt) Does there exist a collection of periodic orbits whose corresponding periodic measures are not weak* dense but they are sufficient for Livshits?
(4) (J. Marshall Reber) Can you improve the theorem to other contexts, i.e. partially hyperbolic systems which are sufficiently nice or matrix valued cocycles which are sufficiently nice?