COHOMOLOGICAL EQUATIONS

JAMES MARSHALL REBER

Contents

1. Motivating Examples 1
1.1. Time Changes of Flows 1
1.2. Stability of Hyperbolic Toral Automorphisms 3
1.3. Finding Invariant Measures 3
2. What Are Cohomological Equations 4
2.1. Definitions and some results 4
2.2. Generalization 10
3. The Livschitz Theorem 10
3.1. Review of Anosov Closing Lemma and Idea of Livschitz Theorem 10
4. Livschitz Theorem 12
4.1. Time change and Orbit Equivalence 14
References 16

1. Motivating Examples

We follow [1] Chapter 2.2 and 2.6, as well as [1] Chapter 5.1 and [3].

1.1. Time Changes of Flows.

We follow [1] Chapter 2.2.

Throughout, M and N will be smooth manifolds. Recall that two flows $\varphi^t : M \to M$ and $\psi^t : N \to N$ are said to be C^m **flow equivalent** (where $m \leq r$) if there is a C^m diffeomorphism $h : M \to N$ such that for all $t \in \mathbb{R}$ we have

$$\varphi^t = h \circ \psi^t \circ h^{-1}.$$

However, there is another way to create equivalence between flows, namely equivalence between orbits of flows. This gives rise to the idea of a time change.

An orbit ψ^t is said to be a **time change** of another flow φ^t if, for each $x \in M$, we have that the orbits coincide and the orientations given by the change of t in the positive direction are the same. In other words, for ψ^t to be a time change of φ^t, we need there to be some function $\alpha(t, x) : \mathbb{R} \times M \to \mathbb{R}$ for which

$$\psi^t x = \varphi^{\alpha(t,x)} x$$

for every $x \in M$. Due to properties of flows, we get a few properties on α, namely the following:

(1) Notice that

$$\psi^0 x = x = \varphi^{\alpha(0,x)} x,$$

so $\alpha(0, x) = 0$.

(2) Notice that

$$\varphi^{\alpha(t+s,x)} x = \psi^{t+s} x = \psi^t \circ \psi^s x.$$

We have

$$\psi^t \circ \psi^s x = \varphi^{\alpha(t,\psi^s x)} \circ \psi^s x = \varphi^{\alpha(t,\psi^s x)} \left(\varphi^{\alpha(s,x)} x\right) = \varphi^{\alpha(t,\psi^s x)+\alpha(s,x)} x.$$

This tells us that

$$\alpha(t, \psi^s x) + \alpha(s, x) = \alpha(t + s, x).$$

(3) By definition, we have

$$\psi^{-t} x = \varphi^{\alpha(-t,x)}(x).$$

Recall

$$\psi^{-t} x = (\psi^t)^{-1} x,$$

so

$$\psi^t \circ \psi^{-t} x = x = \varphi^{\alpha(t,\psi^{-t}x)}(\psi^{-t} x) = \varphi^{\alpha(t,\psi^{-t}x)+\alpha(-t,x)} x.$$

This tells us that

$$\alpha(t, \psi^{-t} x) + \alpha(-t, x) = 0,$$

so

$$\alpha(-t, x) = -\alpha(t, \psi^{-t} x).$$

(4) Since we require that $\alpha(t, x)$ preserves orientation, we have $\alpha(t, x) \geq 0$ for $t \geq 0$.

A function $\alpha(t, x)$ which satisfies these properties is called an **untwisted one-cocycle** over φ^t. These properties will come up later when we wish to describe cocycles in a more general format. We claim that property (2) is actually the real motivation here; that is, we are looking for $\alpha : \mathbb{R} \times M \to \mathbb{R}$ which satisfies the condition that for all s we have

$$\alpha(s, x) = \alpha(t + s, x) - \alpha(t, \psi^s x)$$

for all t. Assuming condition (2), we get condition (1) since

$$\alpha(0, x) = \alpha(t, x) - \alpha(t, \psi^0 x) = \alpha(t, x) - \alpha(t, x) = 0,$$

and we get condition (3) since for $t = 0$ we have

$$\alpha(-s, x) = \alpha(-s, x) - \alpha(0, \psi^{-s} x) = -\alpha(0, \psi^{-s} x).$$

So really a one-cocycle is a function $\alpha : \mathbb{R} \times M \to \mathbb{R}$ which satisfies the condition

$$\alpha(s, x) = \alpha(t + s, x) - \alpha(t, \psi^s x)$$

for all t and which satisfies $\alpha(t, x) \geq 0$ for $t \geq 0$.

One natural question which arises is whether this is an equivalence relation.

Claim. The relation $\psi \sim \varphi$ iff ψ is a time change of φ is an equivalence relation.

Proof. Walk through the definition of an equivalence relation and the definition of a time change. □

Consequently, we then want to consider flows up to time change, and what complexity is preserved by time changes. The first (and maybe easiest) property is that fixed points are preserved.

Claim. If $\psi \sim \varphi$, then the fixed points of ψ coincide with the fixed points of φ.

Proof. The orbit of a fixed point is the fixed point. □

Notice that, by a consequence of this, if $\alpha(t, x) = 0$ for some t and x, then $\alpha(s, x) = 0$ for all $s \in \mathbb{R}$. Similarly, if $\alpha(t, x) \neq 0$, then $\alpha(s, x) > 0$ for all $s > 0$.

Another interesting property of time changes is that if x is not a fixed point, then we have that α is a C^r function with respect to both variables.'

Claim. If $\varphi \sim \psi$ and x is not a fixed point, then $\alpha(t, x)$ is a C^r function in both variables.

Proof. Use the implicit function theorem. □

Already, there are some interesting properties for α, but the question now is how does one find such an α? This is where cohomological equations comes in.

1.2. **Stability of Hyperbolic Toral Automorphisms.** We follow [1] Chapter 2.6.

In [1] **Theorem 2.6.1**, we showed the following result:

Theorem. Any hyperbolic linear automorphism F_L of the two-torus is a factor of any homeomorphism g in the same homotopy class via a uniquely determined semiconjugacy homotopic to the identity. If g is C^0-close to F_L, then the semiconjugacy is close to the identity in the C^0 topology.

The trick to proving this boiled down to an observation that, if such an h exists, we need it to satisfy

$$\overline{h} = L^{-1}\overline{g} + L^{-1} \circ \overline{h} \circ (L + \overline{g}).$$

Breaking it down, we wrote

$$\overline{h} = h_1 e_1 + h_2 e_2, \qquad \overline{g} = g_1 e_1 + g_2 e_2,$$

and we could express it as

$$h_1 = \lambda_1^{-1} g_1 + \lambda_1^{-1} h_1 \circ (L + \overline{g}),$$
$$h_2 = \lambda_2^{-1} g_2 + \lambda_2^{-1} h_2 \circ (L + \overline{g}).$$

To solve this, we created some linear operator and then used a contracting map principle to find a unique solution to it. This ended up being an equation in the form of a coboundary equation. Consequently, using cohomological equation methods, we could solve it that way as well

1.3. **Finding Invariant Measures.** We follow [1] Chapter 5.1 and [3].

Imagine we have M is a compact, smooth oriented manifold and T is a diffeomorphism. This gives rise to a dynamical system (M, T). Using some basic results (Banach-Alaoglu), we have that there exists an invariant measure with respect to T. In general, this measure need not have any interesting structure associated to it. Suppose we have Ω some invariant volume form. One thing we may want out of the measure μ that we find is that it preserves the volume form. If $\rho : M \to \mathbb{R}_{\geq 0}$ represents a density function for our measure μ, then this can be translated to us wanting $\rho\Omega$ is invariant under T.

There are two major concepts we need to review to study this. First, the condition that an n-form is invariant under a function T means that $T^*\omega = \omega$ (the pushforward of the form is the form). So going to the above description, we wish to find ρ so that

$$T^*(\rho\Omega) = (\rho \circ T^{-1})\Omega = \rho\Omega.$$

Next, if $f : M \to M$ is a function, the Jacobian of that function is given by

$$f^*\Omega = (Jf)\Omega.$$

So we can rewrite the above as

$$(\rho \circ T^{-1})\Omega = T^*(\rho\Omega) = (JT)\rho\Omega = \rho\Omega.$$

In other words, we wish to solve the equation

$$\rho \circ T = \frac{\rho}{JT}.$$

Taking logarithms gives us

$$\log(\rho \circ T) = \log(\rho) - \log(JT),$$

which can then be rewritten as

$$\log(JT) = \log(\rho) - \log(\rho \circ T).$$

It turns out that this is given in the form of a cohomological equation. The solution of such an equation can be found using techniques from cohomological equation. Generally we can always find a solution ρ to this equation, but the question of what kind of properties ρ exhibits is an issue when we do the naieve thing (which is similar to the case of looking at general cohomological equations).

2. What Are Cohomological Equations

2.1. Definitions and some results.

This section will follow [1] Chapter 2.9.

A **cohomological equation** is one of the form

$$g(x) = \lambda\varphi(f(x)) - \varphi(x), \tag{1}$$

where g, λ, and f are known, and φ is some unknown scalar function. One can also view it in terms of vector equations, which will have the same format. The goal is to figure out what φ is.

A **one-cocycle** twisted by a representation $\rho : G \to \mathrm{GL}(k, \mathbb{R})$ (here G represents time, so $G = \mathbb{N}, \mathbb{Z}$, or $\mathbb{R}$) for a dynamical system $T : G \times X \to X$ is a map $\alpha : G \times X \to \mathbb{R}^k$ which satisfies

$$\alpha(g_1 + g_2, x) = \rho(g_1)\alpha(g_2, T(g_1)x) + \alpha(g_1, x).$$

If ρ is the identity representation, then α is called an **untwisted cocycle**.

We note a nice property for one-cocycles. Namely, we have the following.

Claim. If α is a one-cocycle, then $\alpha(0, x) = 0$.

Proof. We have

$$\alpha(0, x) = \rho(0)\alpha(0, T(0)x) + \alpha(0, x) = 2\alpha(0, x),$$

and solving gives $\alpha(0, x) = 0$. □

Next, we note that any function $\varphi : X \to \mathbb{R}^k$ gives us a cocycle via setting

$$\alpha(g, x) := \rho(g)\varphi(T(g)x) - \varphi(x).$$

We check this:

$$\begin{aligned} \rho(g_1)\alpha(g_2, T(g_1)x) &= \rho(g_1)\left(\rho(g_2)\varphi(T(g_2)x) - \varphi(T(g_1)x)\right), \\ \alpha(g_1, x) &= \rho(g_1)\varphi(T(g_1)x) - \varphi(x), \end{aligned}$$

and

$$\begin{aligned} \rho(g_1)\alpha(g_2, T(g_1)x) + \alpha(g_1, x) &= \rho(g_1)\left(\rho(g_2)\varphi(T(g_2)x) - \varphi(T(g_1)x)\right) + \rho(g_1)\varphi(T(g_1)x) - \varphi(x) \\ &= \rho(g_1 + g_2)\varphi(T(g_2)x) - \rho(g_1)\varphi(T(g_1)x) + \rho(g_1)\varphi(T(g_1)x) - \varphi(x) \\ &= \rho(g_1 + g_2)\varphi(T(g_2)x) - \varphi(x) = \alpha(g_1 + g_2, x). \end{aligned}$$

Cocycles $\alpha(g, x)$ which correspond to functions φ are called **coboundaries**. Two cocycles are **cohomologous** if their difference is a boundary.

We now examine the discrete time case ($G = \mathbb{N}, \mathbb{Z}$).

Claim. Every cocycle $\alpha(g, x)$ is determined by the function $a(x) := \alpha(1, x)$.

Proof. We claim that if we know the function $a(x)$, we can recover $\alpha(n, x)$ for any n. To see this, notice that if $G = \mathbb{N}$, then it's just a matter of induction. Notice that for $n = 2$ we have

$$\alpha(2, x) = \alpha(1 + 1, x) = \rho(1)\alpha(1, Tx) + \alpha(1, x) = \rho(1)a(Tx) + a(x).$$

The claim then is that

$$\alpha(n, x) = \sum_{i=0}^{n-1} \rho(i)a(T(i)x).$$

Assume it holds for $n - 1$. We can write

$$\alpha(1 + (n - 1), x) = \rho(1)\alpha(n - 1, Tx) + \alpha(1, x).$$

By the induction hypothesis,

$$\rho(1)\alpha(n-1,Tx) = \sum_{i=0}^{n-2} \rho(i+1)a(T(i+1)x).$$

So

$$\alpha(n,x) = \sum_{i=0}^{n-1} \rho(i)\alpha(T(i)x). \tag{2}$$

For $G = \mathbb{Z}$, we use the property $\alpha(0,x) = 0$ for all $x \in X$ to determine what $\alpha(-n,x)$ should be. Notice

$$0 = \alpha((-n)+n,x) = \rho(-n)\alpha(n,T(-n)x) + \alpha(-n,x).$$

Notice

$$\rho(-n)\alpha(n,T(-n)x) = \sum_{i=0}^{n-1} \rho(i-n)a(T(i)T(-n)x) = \sum_{i=1}^{n} \rho(-i)a(T(-i)x),$$

giving us

$$\alpha(-n,x) = \sum_{i=1}^{n} \rho(-i)a(T(-i)x). \tag{3}$$

□

A **coboundary** is a cocycle satisfying the property that for some $\varphi : X \to \mathbb{R}$, we have

$$\alpha(g,x) = \rho(g)\varphi(T(g)x) - \varphi(x).$$

We claim that solving a cohomological equation is equivalent to showing a cocycle is a coboundary. Assume we have that α is a cocycle determined by $g : X \to \mathbb{R}$. If there is a solution to the cohomological equation

$$\alpha(1,x) = g(x) = \lambda\varphi(T(x)) - \varphi(x),$$

where $\lambda = \rho(1)$, then we see that

$$\alpha(n,x) = \sum_{i=0}^{n-1} \rho(i)g(T(i)x),$$

$$\rho(i)g(T(i)x) = \rho(i)\rho(1)\varphi(T(i+1)x) - \rho(i)\varphi(T(i)x),$$

and so the sum will be

$$\alpha(n,x) = \rho(n)\varphi(T(n)x) - \varphi(x)$$

since it's an alternating sum. A similar calculation can be done for $n < 0$, giving us that α is a coboundary. Likewise, if we know α is a coboundary, then we know there is a solution to the cohomological equation given by

$$\alpha(1,x) = g(x) = \rho(1)\varphi(T(x)) - \varphi(x).$$

Remark. I've been assuming the codomain for our functions are scalar valued, but nothing changes when I switch to vector valued (except that we have vector valued functions).

We now need to differentiate between the case of $\rho(1)$ being hyperbolic and $\rho(1)$ being nonhyperbolic (here, hyperbolicity comes in when we interpret this as a vector valued problem). One great thing about hyperbolicity comes from the following theorem.

Theorem. Suppose $\rho(1)$ is hyperbolic. Then for bounded g, the equation

$$\alpha(1,x) = g(x) = \rho(1)\varphi(T(x)) - \varphi(x)$$

admits a unique solution φ. If g is continuous, the solution φ will also be continuous.

Proof. The proof of this should be familiar. We deconstruct our space into expanding and contracting parts, and then use a contraction mapping principle argument to conclude. Let $R = \rho(1)$ for notational simplicity. By hyperbolicity, we can decompose R into R_+ and R_- (expanding and contracting parts, respectively). We obtain two equations from this:

$$a_+(x) = R_+\varphi_+(T(x)) - \varphi_+(x),$$

$$a_-(x) = R_-\varphi_-(T(x)) - \varphi_-(x).$$

We can (by **Proposition 1.2.2**) find a norm with the property that

$$\|R_-\|, \|R_+^{-1}\| < 1.$$

Like in the case of hyperbolic stability, we can use a sort of alternating series type solution:

$$\varphi_+(x) = \sum_{k=1}^{\infty} R_+^{-k} a_+(T(-k)(x)), \qquad \varphi_-(x) = -\sum_{k=0}^{\infty} R_-^{k} a_-(T(k)(x)).$$

Boundedness and continuity follow. For uniqueness, let φ, θ be two solutions. Notice that their difference $\kappa = \varphi - \theta$ solves the equation

$$a(g,x) = \rho(g)\kappa(T(g)x) - \kappa(x)$$

for $\alpha(g,x) = 0$ (the zero one-cycle). So

$$\alpha(1,x) = 0 = \rho(1)\kappa(f(x)) - \kappa(x) \implies \rho(1)\kappa(f(x)) = \kappa(x).$$

Since $\rho(1)$ is hyperbolic, the only *bounded* solution to this will be $\kappa = 0$. □

Remark. We really don't get more structure beyond continuity. Even if we assume α is smooth, f is smooth, and ρ is smooth, φ cannot be expected to be smooth. Generally, the best one can expect is Lipschitz continuous.

We now jump into the non-hyperbolic case. Take, for example, the untwisted cocycle case: $\rho(1) = \mathrm{Id}$. In this case, looking at equations (2) and (3), we have

$$\alpha(n,x) = \sum_{i=0}^{n-1} a(T(i)x), \qquad \alpha(-n,x) = \sum_{i=1}^{n} a(T(-i)x).$$

If we assume x is a periodic point for T, say $T(n)x = x$ for some n,

$$\sum_{i=0}^{n-1} \alpha(T(i)x) = \alpha(n,x) = \varphi(T(n)x) - \varphi(x) = 0.$$

The sum over a periodic orbit is called the **periodic obstruction**. If we ignore any structure on X, then this is not an obstruction at all – we do the obvious thing listed above. If there is some structure we wish to preserve, however. Take for example, consider the dynamical system $T : \mathcal{S}^1 \to \mathcal{S}^1$ which is given by an irrational rotation β. So

$$T(n)x = x + n\beta \pmod 1.$$

The goal is then to solve the cohomological equation

$$g(x) = \varphi(T(1)(x)) - \varphi(x),$$

where g some scalar function. We can find a solution by using the equivalence of cohomological equations to cocycles, so doing the naieve thing gives us the cocycle

$$\alpha(n, x) = \sum_{i=0}^{n-1} g(T(i)x).$$

So we now consider $\mathcal{S}^1/\sim$, where $x \sim y$ if $x \in \mathcal{O}_T(y)$ (two points are equivalent if they are in the same orbit). For each $x \in X$, we have that $x \in [z] \in \mathcal{S}^1/\sim$, so $x = T(n)z$, and we can set

$$\varphi(x) = \varphi(T(n)z) = \sum_{i=0}^{n-1} g(T(i)x).$$

Note that the space $\mathcal{S}^1/\sim$ is the Vitali set, so we're defining our solution φ using a non-measurable set of representatives. This (most likely) leads to a non-measurable solution (which is bad, by all accounts). We can fix this if we know some bounds on α.

Theorem. If $\alpha(n, x)$ is bounded uniformly in n and x, then $a(x) = \varphi(T(1)(x)) - \varphi(x)$ has the solution

$$\varphi(x) = \sup_{n\in\mathbb{N}} \left\{ -\sum_{i=0}^{n} a(T(i)x) \right\}.$$

Moreover the solution is measurable.

Proof. Since it is uniformly bounded, note that φ is well-defined. Plug things in to see it's a solution. $\square$

Remark. We could have used Banach limits to solve this!

Consider the set

$$B_2 = \left\{ (\omega_j) \in \Omega_2 : \forall m, n \in \mathbb{Z}, m > n, \left| \sum_{i=n}^{m} (-1)^{\omega_i} \right| \le 2 \right\}.$$

Recall we have σ_2 is the left shift map;

$$\sigma_2 : \Omega_2 \to \Omega_2, \qquad \sigma_2((\omega_j)) = \omega'_j, \qquad \omega'_j = \omega_{j+1}.$$

First, note that

$$S_2 = \sigma_2|_{B_2} : B_2 \to B_2$$

gives us a topologically transitive system (note it is not topologically mixing – to see this, take (ω_j) which achieves the bound 2). The goal now is to show that the cohomological equation

$$\varphi(S_2\omega) - \varphi(\omega) = (-1)^{\omega_0}$$

has bounded Borel solution, but no continuous solution. This shows that the prior theorem does not guarantee continuity in any sense. We have $g : B_2 \to \mathbb{R}$ is given by $g((\omega_k)) = (-1)^{\omega_0}$. This determines a cocycle

$$\alpha(n, (\omega_k)) = \sum_{i=0}^{n-1} g(S_2^i((\omega_k))) = \sum_{i=0}^{n-1} (-1)^{\omega_i},$$

$$\alpha(-n, (\omega_k)) = \sum_{i=1}^{n} g(S_2^{-i}((\omega_k))) = \sum_{i=1}^{n} (-1)^{\omega_{-i}}.$$

By construction, this cocycle is bounded uniformly in both (ω_k) and n, so we can invoke the prior theorem to get a solution of the form

$$\varphi(x) = \sup_{n\in\mathbb{N}} \left\{ -\sum_{i=0}^{n} (-1)^{\omega_i} \right\}.$$

This is a bounded Borel measurable solution, but not continuous. The fact that continuity is impossible follows by the fact that S_2 is not topologically conjugate to *any* topological Markov chain – the discontinuities in our solution φ will arise from points where the preimage has more than one point in the semiconjugacy (see **Exercise 1.9.11** and **Exercise 1.9.12** in [1]).

We can, however, fix this if we put some more constraints on our system.

Theorem (Gottschalk, Hedlund)**.** Let X be a compact metric space, $f : X \to X$ a minimal[1] continuous map, and $g : X \to \mathbb{R}$ a continuous map such that for all $M > 0$, there is an n so that

$$\left| \sum_{i=0}^{n} g(f^i(x_0)) \right| < \infty$$

for some $x_0 \in X$. Then there is a continuous $\varphi : X \to \mathbb{R}$ such that

$$\varphi \circ f - \varphi = g.$$

Proof. First, note that this series is bounded for *any* point $y \in X$. We proceed by contradiction. Suppose there is some $y \in X$ so that for all $M > 0$, there is an $n > 0$ with the property that

$$\left| \sum_{i=0}^{n} g(f^i(y)) \right| > M.$$

By continuity, this implies that for all x sufficiently close to y, we also have

$$\left| \sum_{i=0}^{n} g(f^i(x)) \right| > M.$$

By the density of the orbit of x_0, we get $f^{n_0}(x_0) = x$ for some n_0. Hence

$$\sup_{n\in\mathbb{N}} \left| \sum_{i=0}^{n} g(f^i(x_0)) \right| \geq \left| \sum_{i=0}^{n} g(f^{i+n_0}(x_0)) \right| > M.$$

This contradicts the fact that the series on the left is bounded. Now if we take the cocycle generated by g, say $\alpha(n, x)$, this says that the cocyle is uniformly bounded with respect to both variables, so we can apply the last theorem to find that there is a solution

$$\varphi(x) = \sup_{n\in\mathbb{N}} \left\{ -\sum_{i=0}^{n} g(f^{-i}(x)) \right\}.$$

We define the **oscillation** of a function $\psi : X \to \mathbb{R}$ at a point x to be

$$\mathrm{Osc}_\psi(x) := \lim_{\delta\to 0} \left(\sup\{\psi(y) : |x-y| < \delta\} - \inf\{\psi(y) : |x-y| < \delta\} \right).$$

Note that oscillation vanishing is equivalent to continuity. The first observation here is that for $\epsilon > 0$ the set

$$E_{\epsilon,\varphi} = \{x \in X : \mathrm{Osc}_\varphi(x) \geq \epsilon\}$$

is closed. Notice that

$$\mathrm{Osc}_g(x) = 0 \text{ for all } x \in X.$$

[1] This just means that the orbit of every point is dense.

Using the relation

$$g(x) = \varphi(f(x)) - \varphi(x),$$

we see that $E_{\epsilon,\varphi}$ is f-invariant. Since $E_{\epsilon,\varphi}$ is a closed, f-invariant set, and f is minimal, we get that it either must be X or empty. The goal now is to show for $\epsilon > 0$ the set must be empty.

Notice φ is the pointwise limit of the function

$$\varphi_k(x) = \sup_{n \leq k} \left\{ -\sum_{i=0}^{n} g(f^{-i}(x)) \right\},$$

which is continuous. Consider

$$O_{\epsilon,k} = \{x \in X : |\varphi - \varphi_k| \leq \epsilon/2\}.$$

Notice that for $x \in O_{\epsilon,k}$, we have $\mathrm{Osc}_\varphi(x) \leq \epsilon$. This is since

$$\varphi_k(x) - \epsilon/2 \leq \varphi(x) \leq \epsilon/2 + \varphi_k(x),$$

$$\begin{aligned} \mathrm{Osc}_\varphi(x) = \lim_{\delta \to 0} (\sup\{\varphi(y) : |x-y| < \delta\} - \inf\{\varphi(y) : |x-y| < \delta\}) \\ \leq \epsilon + \mathrm{Osc}_{\varphi_k}(x) = \epsilon. \end{aligned}$$

Notice that $O_{\epsilon,k}$ is nonempty for all $\epsilon > 0$ and k sufficiently large, so therefore $E_{\epsilon,\varphi}$ must be zero for $\epsilon > 0$. □

Claim. Let $f : X \to X$ be a topologically transitive homeomorphism of a compact metric space and g a continuous function on X. Any two solutions of

$$\varphi(f(x)) - \varphi(x) = g(x)$$

differ by a constant.

Proof. By **Corollary 1.4.4** in [1], there are no f-invariant nonconstant continuous functions $\kappa : X \to \mathbb{R}$. Let $\kappa(x) = \varphi(x) - \psi(x)$. We see this is f-invariant, since

$$\kappa(f(x)) = \varphi(f(x)) - \psi(f(x)) = \varphi(x) + g(x) - \psi(x) - g(x) = \varphi(x) - \psi(x) = \kappa(x).$$

Invoking the corollary, we get that $\kappa : X \to \mathbb{R}$ is a constant function, so $\kappa(x) = t$. Thus $\varphi = t + \psi$. □

Using the two prior results, we get the following.

Corollary. Let X be a compact metric space, $f : X \to X$ a minimal continuous map, and $g : X \to \mathbb{R}$ a continuous map such that for all $M > 0$, there is an n so that

$$\left| \sum_{i=0}^{n} g(f^i(x_0)) \right| < \infty$$

for some $x_0 \in X$. Then there is a continuous $\varphi : X \to \mathbb{R}$ such that

$$\varphi \circ f - \varphi = g.$$

Such a function is unique up to an additive constant.

2.2. **Generalization.** In the above, we mostly focused on the case of a discrete group. We generalize these notions now, following [2].

We consider $(G, \cdot)$ a group (generally at least a topological group), $(U, +)$ an abelian group (also generally at least a topological group), and (X, f) a dynamical system. A **cocycle** is a map $\alpha : G \times X \to U$ which satisfies the **cocycle** equation:

$$\alpha(g \cdot h, x) = \alpha(h, x) + \alpha(g, hx).$$

It turns out that this general notion is useful for building more complex dynamical systems out of old; for example, we can use the cocycle to create a skew product $X \times_\alpha U$ which as a set is

$$\{(x, u) : x \in X, u \in U\},$$

and has the group action

$$g(x, u) = (gx, u + \alpha(g, x)).$$

One example is to consider the dynamical system (X, f) and give it a flow under a function (sometimes suspension flow?) You observe a measurable function $T : X \to (0, \infty)$ and consider

$$\Gamma_f = \{(x, t) : x \in X, 0 \leq t < T(x)\},$$

$$S_n(x) = \begin{cases} \sum_{i=0}^{n-1} T(f^i(x)) \text{ if } n \geq 1 \\ 0 \text{ if } n = 0 \\ -\sum_{i=1}^{n} T(f^{-i}(x)) \text{ if } n < 0, \end{cases}$$

$$n(x, t, s) = \min\{k \in \mathbb{Z} : s + t < S_{k+1}(x)\}.$$

Then the flow defined by

$$T_f^s(x, t) = (T^{n(x,t,s)}(x), s + t - S_{n(x,t,s)}(x))$$

gives us a cocycle (note: this is a technical thing to check). You can make T and f have whatever properties you desire in this setting (for example, measurable).

Another technical example given by [2] is the construction of dynamical systems on nilmanifolds (beyond my scope of understanding).

A **coboundary** is a special cocycle satisfying the property that for some $f : X \to U$ we have

$$\alpha(g, x) = f(gx) - f(x).$$

The relevance in the above setting is that the extension of a cobundary can be conjugated to a trivial extension.

The idea is that we want to measure the failure of cocycles to be coboundary terms. We can introduce the first cohomology group by setting $B^1(G, X, U)$ to be all of the coboundaries, $Z^1(G, X, U)$ to be all of the cocycles, and $H^1(G, X, U) = Z^1(G, X, U)/B^1(G, X, U)$ (one must check these spaces are abelian groups, but they are).

3. The Livschitz Theorem

3.1. **Review of Anosov Closing Lemma and Idea of Livschitz Theorem.** The final goal of these notes is to prove the **Livschitz theorem** (see **Theorem 19.2.1** [1]). To do so, we need to review some facts about the **Anosov Closing Lemma** (see **Theorem 6.4.15** [1]).

Suppose M is a smooth manifold, $U \subseteq M$ an open subset, $f : U \to M$ a C^1 diffeomorphism onto its image, and $\Lambda \subseteq U$ a compact f-invariant set. We say that the set Λ is a **hyperbolic set for** f if there is a Riemannian metric (called the **Lyapunov metric**) in an open neighborhood U of Λ and $\lambda < 1 < \mu$ such that for each $x \in \Lambda$ the sequence of differentials

$$(Df)_{f_x^n} : T_{f_x^n} M \to T_{f_x^{n+1}} M, \qquad n \in \mathbb{Z}$$

admits a (λ, μ) splitting. Recall a (λ, μ) **splitting** means that the space $T_{f^n_x} M$ decomposes into

$$T_{f^n_x} M = E^+_n \oplus E^-_n$$

with the property that $(Df)_{f^n_x}(E^{\pm}_n) = E^{\pm}_{n+1}$ and

$$\left\| (Df)_{f^n_x}|_{E^-_n} \right\| \leq \lambda, \qquad \left\| (Df)^{-1}_{f^n_x}|_{E^+_{n+1}} \right\| \leq \mu^{-1}.$$

A periodic sequence of points $x_0, x_1, \ldots, x_{n-1}, x_n = x_0$ is a **periodic ϵ-orbit** or a **periodic pseudo orbit** if

$$d(f(x_k), x_{k+1}) < \epsilon \text{ for } k = 0, \ldots, m-1.$$

We now have the appropriate vocab to discuss the Anosov closing lemma.

Theorem (Anosov Closing Lemma)**.** Let Λ be a hyperbolic set for $f : U \to M$. There exists an open neighborhood V with $\Lambda \subseteq V$, C a constant, and $\epsilon_0 > 0$ so that for any $\epsilon < \epsilon_0$ and any periodic ϵ-orbit $(x_0, \ldots, x_m) \subseteq V$ there is a point $y \in U$ such that $f^m(y) = y$ and

$$d(f^k(y), x_k) < C\epsilon \text{ for } k = 0, \ldots, m-1.$$

Sketch of proof. The proof of this is to decompose your space nicely (since things are hyperbolic, we can do this) and use the coordinates on these subspaces. On these subspaces we can view things as small perturbations of hyperbolic linear maps, and so we use the fact that these are nice to create an operator whose fixed points are the y which give us the result. We note that if we choose C and ϵ_0 appropriately, then the operator is a contraction, so has a unique fixed point. You then check that the fixed point stays inside of the desired neighborhood. □

Note that we can actually strengthen the statement.

Theorem (Amplification of Anosov Closing Lemma)**.** Let Λ be a hyperbolic set with a (λ, μ)-splitting for $f : U \to M$. Then for any $\alpha \geq \max(\lambda, \mu^{-1})$, there exists a $\delta > 0$ and a $C > 0$ such that if $x \in \Lambda$, $y \in U$, and

$$d(f^k(y), f^k(x)) < \delta \text{ for } k = 0, \ldots, n,$$

then in fact

$$d(f^k(y), f^k(x)) < C\alpha^{\min\{k, n-k\}} \cdot (d(x, y) + d(f^n(x), f^n(y))).$$

Sketch of proof. Use a localization procedure. □

The idea behind the Livschitz theorem is as follows: we have a cohomological equation we wish to solve, say

$$\varphi \circ f - \varphi = g.$$

The naieve thing to do here is to just find a point with dense orbit and set

$$\varphi(f^n(x)) = \sum_0^{n-1} g(f^i(x)).$$

This gives us "almost every point" (not in a measure theoretic sense but in a topological sense). We now want to somehow extend our solution from this dense subset to the whole space. However, we already know that there are periodic obstructions, so this somehow determines the function on periodic points (the sums need to vanish). There also may hidden data which we do not know anything about, but may have wildly different behavior. It turns out that if we have Hölder data (meaning g is Hölder continuous) on the hyperbolic set, then the data for φ is completely determined by the data on the periodic orbits (much of this discussion is vague; the conversation can become a little more rigorous after reading **Chapter 18** [1]). This is the content of Livschitz's theorem.

4. Livschitz Theorem

Theorem (Livschitz theorem)**.** Let M be a Riemannian manifold, $U \subseteq M$ open, $f : U \to M$ a smooth embedding, $\Lambda \subseteq U$ a compact topologically transitive hyperbolic set, and $g : \Lambda \to \mathbb{R}$ Hölder continuous. Suppose that for every $x \in \Lambda$ such that $f^n(x) = x$ (i.e. every periodic point), we have

$$\sum_0^{n-1} g(f^i(x)) = 0.$$

Then there is a continuous $\varphi : \Lambda \to \mathbb{R}$ such that

$$g = \varphi \circ f - \varphi.$$

Moreover, φ is unique up to an additive constant and Hölder with the same exponent as g.

Notice that this is a great achievement in the context of **Section 2** – we not only are guaranteed a unique (up to an additive constant) solution, but this solution is also guaranteed to have a great deal of structure, namely Hölder continuous.

Proof. First, note that Λ is a topologically transitive hyperbolic set. This means that $f|_\Lambda$ is topologically transitive. This means we have $x_0 \in \Lambda$ so that the orbit $\mathcal{O}_f(x_0)$ is dense in Λ. Now choose $\varphi(x_0) \in \mathbb{R}$. Letting

$$\alpha(n, x) = \begin{cases} \sum_0^{n-1} g(f^i(x)) \text{ if } n \geq 0 \\ -\sum_1^n g(f^{-i}(x)) \text{ if } n < 0, \end{cases}$$

we see that

$$\varphi(f^n(x_0)) = \varphi(x_0) + \alpha(n, x_0).$$

We now need to check that φ on $\mathcal{O}_f(x_0)$ is Hölder continuous. In fact, we will show that it has the same exponent as g.

Claim. The function

$$\varphi(f^n(x_0)) = \varphi(x_0) + \alpha(n, x_0)$$

on $\mathcal{O}(x_0)$ is Hölder continuous with the same exponent as g.

Proof. Choose $n, m \in \mathbb{N}$ large enough so that

$$\epsilon := d(f^n(x_0), f^m(x_0))$$

is small enough to apply the amplification theorem and the Anosov closing lemma. The Anosov closing lemma gives us $C > 0$, $y = f^{m-n}(y)$, and the amplification theorem gives us $\beta \in (0, 1)$ so that

$$d(f^{n+i}(x_0), f^i(y)) \leq C\epsilon\beta^{\min\{i, m-n-i\}}.$$

Now we assumed that g was Hölder continuous, so suppose it is Hölder continuous with exponent $\gamma \in (0, 1]$. Then there exists $M > 0$ so that if $d(x_1, x_2)$ is sufficiently small, then

$$|g(x_1) - g(x_2)| \leq Md(x_1, x_2)^\gamma.$$

We now just plug things in. We use the definition to get

$$|\varphi(f^n(x_0)) - \varphi(f^m(x_0))| \leq \left| \sum_0^{m-n-1} g(f^{n+i}(x_0)) \right|.$$

We can add and subtract $\sum_0^{m-n-1} g(f^i(y))$ and use the fact that this is a periodic point to get

$$\left| \sum_0^{m-n-1} g(f^{n+i}(x_0)) \right| = \left| \sum_0^{m-n-1} (g(f^{n+i}(x_0)) - g(f^i(y))) \right|.$$

Apply the triangle inequality to get

$$\left|\sum_{0}^{m-n-1}(g(f^{n+i}(x_0))-g(f^i(y)))\right| \leq \sum_{0}^{m-n-1}\left|(g(f^{n+i}(x_0))-g(f^i(y)))\right|.$$

Now we use the two above facts;

$$\left|(g(f^{n+i}(x_0))-g(f^i(y)))\right| \leq Md(f^{n+i}(x_0),f^i(y))^\gamma \leq MC^\gamma\epsilon^\gamma\beta^{\gamma\min\{i,m-n-i\}}.$$

We substitute this in, giving us

$$\sum_{0}^{m-n-1}\left|(g(f^{n+i}(x_0))-g(f^i(y)))\right| \leq \sum_{0}^{m-n-1} MC^\gamma\epsilon^\gamma\beta^{\gamma\min\{i,m-n-i\}}.$$

Notice

$$\beta^{\gamma\min\{i,m-n-i\}} \leq \beta^{\gamma i},$$

so

$$\sum_{0}^{m-n-1} MC^\gamma\epsilon^\gamma\beta^{\gamma\min\{i,m-n-i\}} \leq MC^\gamma\epsilon^\gamma\sum_{0}^{m-n-1}\beta^{\gamma i}.$$

This is now a geometric series, so in particular we have

$$MC^\gamma\epsilon^\gamma\sum_{0}^{m-n-1}\beta^{\gamma i} < MC^\gamma\epsilon^\gamma\frac{1}{1-\beta^\gamma}.$$

Recall how we chose ϵ though; we have

$$\epsilon = d(f^n(x_0),f^m(x_0)).$$

Substituting this in, we have the final result:

$$|\varphi(f^n(x_0))-\varphi(f^m(x_0))| < \frac{MC^\gamma}{1-\beta^\gamma}(d(f^n(x_0),f^m(x_0)))^\gamma.$$

□

So φ is uniformly continuous on $\mathcal{O}(x_0)$, and we can extend it to a continuous function on Λ which shares the same Hölder exponent. The uniqueness is the same argument as in the Gottschalk-Hedlund theorem – the difference of two solutions is a continuous f-invariant function, so constant. Since g and $\varphi\circ f-\varphi$ coincide on a dense set and are continuous, they must agree on the entire set, so we have this is indeed a solution to the cohomological equation. □

As noted in [4], it turns out that if we replace $\mathbb{R}$ with any *complete* topological group with a bi-invariant metric, then the theorem still holds. However, if we tried replacing $\mathbb{R}$ with $\mathrm{GL}_n(\mathbb{R})$, then the proof falls apart. The idea is that the failure of commutativity really makes things harder when trying to prove things are close to the identity.

Recall the Spectral Decomposition Theorem (**Theorem 18.3.1** [1]).

Theorem (Spectral Decomposition Theorem)**.** Let M be a Riemannian manifold with $U \subseteq M$ open, $f : U \to M$ a diffeomorphism, and $\Lambda \subseteq U$ a compact locally maximal hyperbolic set for f. Then there exist disjoint closed sets $\Lambda_1,\ldots,\Lambda_m$ and a permutation $\sigma \in \mathrm{Sym}(m)$ such that

$$NW(f|_\Lambda) = \bigsqcup_{i=1}^m \Lambda_i, \qquad f(\Lambda_i) = \Lambda_{\sigma(i)},$$

and when $\sigma^k(i) = i$ then $f^k|_{\Lambda_i}$ is topologically mixing.

Using this, we can drop the assumption of transitivity.

Corollary. Let Λ be a compact locally maximal hyperbolic set for $f : U \to M$ and $\Lambda' := NW(f|_\Lambda)$. Suppose $\varphi : \Lambda' \to \mathbb{R}$ is Hölder continuous and that for every $x \in \Lambda'$ with $f^n(x) = x$ we have

$$\sum_{i=0}^{n-1} \varphi(f^i(x)) = 0.$$

Then there exists continuous $\Phi : \Lambda' \to \mathbb{R}$ such that $\varphi = \Phi \circ f - \Phi$ and Φ is Hölder continuous with the same exponent as φ. Moreover Φ is unique up to addition of a function that is constant on each topologically transitive component of Λ'.

Proof. Break things up into their respective components, which are topologically transitive. Now apply. $\square$

We can also extend the Livschitz theorem to a C^1 version as well.

Theorem (C^1 Livschitz)**.** Let M be a Riemannian manifold, $f : U \to M$ a smooth embedding with a compact topologically transitive hyperbolic set, and $g : \Lambda \to \mathbb{R}$ a C^1 function. Suppose that for every $x \in M$ such that $f^n(x) = x$ we have the vanishing condition. Then there exists a C^1 function $\varphi : \Lambda \to \mathbb{R}$ such that

$$g = \varphi \circ f - \varphi$$

and φ is unique up to an additive constant.

Proof. Use the usual Livschitz theorem to get a Lipschitz continuous solution. We then show it is C^1 along the unstable and stable leaves. If x and y are close points on a stable leave, then

$$\begin{aligned}\varphi(y) - \varphi(x) &= \lim_{n\to\infty} \left(-\sum_0^n (g(f^i(y)) - g(f^i(x))) + \varphi(f^n(x)) - \varphi(f^n(y)) \right) \\ &= -\sum_0^\infty (g(f^i(y)) - g(f^i(x))).\end{aligned}$$

Keep x fixed and differentiate with respect to $y = x + tv$ at $t = 0$. By the chain rule, this gives

$$D_v\varphi(x) = -\sum_0^\infty D_{v_i}\varphi(f^i(x))D_v(f^i)(x), \qquad v_i = Df^i v.$$

Note that, since v is a stable vector, we have $D_v(f^i)$ is exponentially small. The first factor $D_{v_i}(\varphi(f^i(x)))$ is also exponentially small, since $\varphi \in C^1$ and the v_i are exponentially small. We get that the series converges uniformly, and so to a well-defined continuous function. The same argument applies in the unstable direction. By an earlier result (**Lemma 19.1.10**) we get that φ is C^1. $\square$

4.1. **Time change and Orbit Equivalence.** We recall some notions now.

Two flows φ^t and ψ^t (say on the same manifold M) are said to be *flow equivalent* if there exists a diffeomorphism $h : M \to M$ such that

$$\varphi^t = h \circ \psi^t \circ h^{-1}.$$

Note this is a really strict equivalence. We can loosen it a bit by switching, instead, to orbit equivalence. So two flows φ^t and ψ^t will now be equivalent if, for each $x \in M$, the orbits

$$\mathcal{O}_\varphi(x) \cap \mathcal{O}_\psi(x) = \mathcal{O}_\varphi(x)$$

and the orientations given by the change of t in the positive direction are the same. This is called a **time change**.

Naturally if ψ is a time change of φ, then there is some function $\alpha(t,x)$ so that for every $x \in M$

$$\psi^t(x) = \varphi^{\alpha(t,x)}x$$

The properties we specified about orientation tell us that $\alpha \geq 0$, and we see that we have α is a cocycle by flow properties.

Two flows are φ^t and ψ^t are said to be **orbit equivalent** if there exists a diffeomorphism $h : M \to M$ such that the flow

$$\chi^t = h^{-1} \circ \psi^t \circ h$$

is a time change of the flow φ^t. So the orbits of ψ^t are mapped onto the orbits of φ^t by the diffeomorphism h which preserves the orientation in the positive-time direction.

A time change produces a flow that is flow equivalent to the original one if the conjugating diffeomorphism itself preserves each orbit of the original flow. In other words, we have

$$h(x) = \varphi^{\beta(x)}(x),$$

where β is a differentiable function and its derivative in the direction of the flow

$$(\xi\beta)(x) = \left.\frac{d\beta(\varphi^t(x))}{dt}\right|_{t=0}$$

is positive if $\xi(x) \neq 0$. So

$$(h \circ \varphi^t \circ h^{-1})(x) = \varphi^{\beta(x)+t-\beta(\varphi^t x)}(x),$$

with

$$\alpha(t,x) - t = \beta(x) - \beta(\varphi^t(x)).$$

This should be a familiar formulation at this point. We see that α will thus be a cocycle.

We've said a lot about the Livschitz theorem in the case of discrete G. We want to now consider continuous G. As before, we will need some kind of an Anosov Closing Lemma.

Theorem (Anosov Closing Lemma for Flows)**.** Suppose that for some $x \in X$ and all $t > 0$ we have that $d(x, \varphi^t(x)) < \epsilon$. Then there is a periodic $x_0 \in X$ with $d(x, x_0) \leq C\epsilon$.

The sketch of the idea is to find nice transverse sets around your points and then do the same kind of argument as in the usual Anosov Closing Lemma. I will sketch the proof.

Sketch. We follow the type of argument in []. Use **Proposition 6.4.11** [1] which says that there is an open neighborhood for both transverse manifolds $W^u(x)$ and $W^s(x)$ in $W^s(x) \cap W^u(x)$. Considering small enough ϵ balls, we can find $\epsilon > 0$ so that $D_\epsilon(x) = W^s_\epsilon(x) \cap W^u_\epsilon(x)$ is an open neighborhood which satisfies the condition that

$$\varphi^t(D_\epsilon(x)) \subseteq D_{\epsilon'}(\varphi^t(x)).$$

So we can find a small t_0 with $|t_0| \leq C\epsilon$ so that $g_{t+t_0}(y) \in D_{\epsilon'}(x)$ for all $y \in D_\epsilon(x)$. We can then consider the map $f : D_\epsilon(x) \to D_{\epsilon'}(x)$ defined by $y \mapsto g_{t+t_0}(y)$. Using transversality now, there is a $z \in W^u(x)$ so that $f(z) \in W^s(x)$. Thus

$$d(z,x) = d(\varphi_{-t-t_0}(f(z)), \varphi_{-t-t_0}(f(x))) < Ce^{-t}\epsilon.$$

Now take $x_1 \in W^s(z) \cap W^u(f(z))$. Then

$$d(f(x_1), f(z)) \leq Ce^{-t}\epsilon.$$

Continue this process to build a periodic orbit. □

Proposition 1 (Livschitz for Flows)**.** Let M be a Riemannian manifold, φ^t a smooth flow, $\Lambda \subseteq M$ a compact topologically transitive hyperbolic set, and $g : \Lambda \to \mathbb{R}$ Hölder continuous (or C^1) such that for every periodic point $x = \varphi^T(x)$ we have

$$\int_0^T g(\varphi^t(x))dt = 0.$$

Then there is a Hölder continuous (or C^1) $G : \Lambda \to \mathbb{R}$ (with the same exponent if it's Hölder) such that $g = \frac{\partial G(\varphi^t(x))}{\partial t}\big|_{t=0}$. Furthermore G is unique up to an additive constant.

Proof. Thanks to the vanishing condition, we can create a cocycle. By topological transitivity, we see that $\{\varphi^t(x_0)\}_{t\in\mathbb{R}}$ is dense for some choice of x_0. Consider the cocycle

$$\alpha(t, x_0) = \int_0^t g(\varphi^t(x))dt.$$

One may check this satisfies the cocycle conditions. Select a value for $G(x_0) \in \mathbb{R}$ arbitrarily. We claim that we must have

$$G(\varphi^t(x_0)) = G(x_0) + \alpha(t, x_0).$$

Using the closing lemma, we can now repeat the argument above. We see

$$|G(\varphi^r(x_0)) - G(\varphi^s(x_0))| = \left|\int_0^{r-s} g(\varphi^{s+t}(x_0))dt\right|.$$

Now suppose r, s are such that $d(\varphi^r(x_0), \varphi^s(x_0)) = \epsilon$ is small enough to invoke the Anosov closing lemma (in fact, the amplification applies here as well). Then we find periodic y with period $r - s$ which is close to x_0. Again use the uniform continuity on the periodic orbits to get it on all of Λ. So G is continuous, and by the same argument will have the same Hölder exponent. The same kind of argument for C^1 applies for G as well, where we check on the stable and unstable leaves. □

This theorem is useful for examining time changes.

Proposition 2. Let φ^t be a flow on a manifold M with a compact topologically transitive hyperbolic set Λ and ψ^t a time change of φ^t. If the periods of all periodic orbits of φ^t and ψ^t agree, then φ^t and ψ^t are flow equivalent via a homeomorphism which is Hölder continuous if the time change is and C^1 if the time change is C^1.

Proof. Consider the time change

$$\psi^t(x) = \varphi^{\alpha(t,x)}(x).$$

This arises from a flow equivalence if there is a $\beta : \Lambda \to \mathbb{R}$ such that $\alpha(t, x) - t = \beta(x) - \beta(\varphi^t(x))$. If we let $\gamma(t, x) = \alpha(t, x) - t$, we can check this is a cocycle. Thus we can invoke the Livschitz theorem for flows to find a β with the structure of α. □

Thus we see that orbit equivalence is the same as flow equivalence when the periods of the corresponding periodic orbits are the same.

References

[1] Anatole Katok and Boris Hasselblatt. *Introduction to the Modern Theory of Dynamics Systems.*

[2] Terrence Tao. *Cohomology for dynamical systems.* Link.

[3] Akhil Mathew. *The cohomological equation for dynamical systems.* Link.

[4] Akhil Mathew. *The Livsic theorem on the cohomological equation.* Link.

[5] Alexander Gorodnik. *Lecture Notes for "Dynamics and Quantum Chaos on hyperbolic surfaces".* Link.