A SURVEY OF THE LIVSCHITZ THEOREM

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Contents

1. Introduction

Suppose we have some dynamical system $f : X \to X$, and suppose we have some function $g : X \to \mathbb{R}$. Under what conditions can we say that there exists a solution $\varphi : X \to \mathbb{R}$ to the functional equation

$$g = \varphi \circ f - \varphi? \tag{1}$$

Such equations are called **cohomological equations** and they arise naturally in the study of dynamical systems. Some examples are outlined in Section 19.2 [7]. For example, we can use the theorem in the context of finding smooth invariant measures for Anosov diffeomorphisms, for finding a flow equivalence between smooth flows which shares the same periodic orbit lengths, for establishing spectral rigidty of surfaces, and for establishing the topological stability of hyperbolic toral automorphisms.

Livschitz[1] managed to solve the above problem in the early 1970s[2] for a broad class of systems; see [9] and [10]. The exact statement is as follows.

Theorem 1 (Livschitz (1971), Theorem 19.2.1 [7])**.** Let M be a Riemannian manifold, $f : U \to M$ a smooth embedding with a compact topologically transitive[3] hyperbolic set Λ and $g : \Lambda \to \mathbb{R}$ a

[1]Due to different translations, there are many different ways of spelling his name. I will use Livschitz in this paper, however Livsic, Livshitz, and Livshits are all different ways of spelling it.

[2]Remarkably he was an undergraduate at the time. See here for more on the history.

[3]There exists $x \in \Lambda$ with dense forward orbit.

Hölder continuous function.[4] Suppose that for every $x \in M$ with period n we have

$$\sum_{i=0}^{n-1} g(f^i(x)) = 0.$$

Then there exists a continuous $\varphi : \Lambda \to \mathbb{R}$ such that

$$g = \varphi \circ f - \varphi.$$

Moreover φ is unique up to an additive constant and Hölder with the same exponent as g.

The outline of the article is as follows. In Section 2 we present the necessary preliminaries for both the theorem and the examples, introducing the definitions of cocycles and coboundaries are in the context of dynamical systems, the definition of flow equivalence, the definition of orbit equivalence, and the definition of hyperbolicity along with some results. In Section 3 we prove the theorem and discuss some generalizations. Finally in Section 4 we discuss some of the applications of the theorem, focusing specifically on time changes and orbit equivalences and marked length spectrum rigidity.

2. Preliminaries

2.1. **Cocycles and cohomology.** We follow Section 2.9 [7] as well as some unpublished notes by Dr. Gogolev.

Let $f : X \to X$ be a homeomorphism of a compact metric space X. A **cocycle over** f is a function $\alpha : \mathbb{Z} \times X \to \mathbb{R}$ satisfying the following identity:

$$\alpha(k+n, x) = \alpha(k, f^n(x)) + \alpha(n, x). \tag{2}$$

We call Equation 2 the **cocycle identity.**

> **Problem 1.** For a cocycle as above, show the following properties:
> (1) $\alpha(0, x) = 0$,
> (2) $\alpha(-n, x) = -\alpha(n, f^{-n}(x))$.

We call a function $a : X \to \mathbb{R}$ a **generator** for a cocycle α if $\alpha(1, x) = a(x)$.

Claim 1. There is a bijection between cocycles and functions $a : X \to \mathbb{R}$. In other words, we can recover a cocycle from its generator.

Proof. One direction is clear. For any cocycle $\alpha(n, x)$ we can define a generator $a(x)$ by setting $a(x) := \alpha(1, x)$. For the other direction, take $a : X \to \mathbb{R}$. We can create a cocycle from this by setting

$$\alpha(n, x) = \begin{cases} \sum_{i=0}^{n-1} a(f^i(x)) \text{ if } n \geq 1, \\ 0 \text{ if } n = 0, \\ \sum_{i=1}^{-n} a(f^{-i}(x)) \text{ if } n < 0. \end{cases} \tag{3}$$

We need to check that this actually gives us a cocycle. Assuming $k, n \geq 1$ we have

$$\alpha(k+n, x) = \sum_{i=0}^{k+n-1} a(f^i(x)) = \sum_{i=0}^{n-1} a(f^i(x)) + \sum_{i=0}^{k-1} a(f^{n+i}x) = \alpha(n, x) + \alpha(k, f^n(x)).$$

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[4] We remark that Hölder here slightly deviates from the usual Hölder definition. We say $g : X \to \mathbb{R}$ is Hölder if for every x and y sufficiently close we have $|g(x) - g(y)| \leq Cd(x, y)^\alpha$ for $C, \alpha > 0$.

The argument for other values of k and n follows analogously, so we omit it. Thus for every function $a : X \to \mathbb{R}$ we can associate a cocycle.

Let's check that the above maps are inverses. Suppose we have a cocycle $\alpha(n, x)$ and we associate to it $a(x) := \alpha(1, x)$. Let

$$\beta(n, x) := \sum_{i=0}^{n-1} a(f^i(x)).$$

We check that $\beta = \alpha$. By construction we have $\beta(1, x) = \alpha(1, x)$. Assume that up to $n-1$ we have for all $x \in X$ the relation $\beta(n-1, x) = \alpha(n-1, x)$. Then

$$\beta(n, x) = \beta((n-1)+1, x) = \beta(1, x) + \beta(n-1, f(x)) = \alpha(1, x) + \alpha(n-1, f(x)) = \alpha(n, x).$$

So by induction this holds for $n \geq 1$. By Problem 2, we see that $\alpha(0, x) = 0 = \beta(0, x)$ for all $x \in X$. Now we see that

$$\begin{aligned}\beta(-n, x) + \alpha(n, f^{-n}(x)) = \beta(-n, x) + \beta(n, f^{-n}(x)) = \beta(n + (-n), x) = \beta(0, x) = \alpha(0, x) \\ = \alpha(n + (-n), x) = \alpha(-n, x) + \alpha(n, f^{-n}(x)).\end{aligned}$$

Subtracting $\alpha(n, f^{-n}(x))$ from both sides gives us $\beta(-n, x) = \alpha(-n, x)$ for all $x \in X$. Thus the cocycles are equal. Taking a function $a : X \to \mathbb{R}$, we create the cocycle $\alpha(n, x)$ as described earlier. It's clear that the function $\alpha(1, x) = a(x)$, giving us the bijection. □

Problem 2. Let $f : S^1 \to S^1$ be an orientation preserving circle diffeomorphism. Prove that $\alpha(n, x) = \log((f^n)')(x)$ is a cocycle over f.

Problem 3. Let α, β be two cocycles.

(1) Show that the sum of two cocycles is still a cocycle.
(2) Show that scaling a cocycle still gives you a cocycle. In other words, we have that cocycles form a real vector space.
(3) Let $h : X \to X$ be a homeomorphism such that $h \circ f = f \circ h$ and $h \circ f^{-1} = f^{-1} \circ h$. Show $\alpha(t, h(x))$ is a cocycle.

The above is a specific example of a more general theory. Let $f : X \to X$ be a homeomorphism of a compact metric space X, and let G be a topological group. A **G-valued cocycle over** f is a function $\alpha : \mathbb{Z} \times X \to G$ which satisfies the following identity:

$$\alpha(k + n, x) = \alpha(k, f^n(x)) \cdot \alpha(n, x).$$

We also assume G is commutative, though it's not really necessary.

Problem 4. Do all of the above properties hold for general G?

We can generalize it even further – this second generalization will be especially useful for flows. If we let Γ be a topological group, we can define a **continuous group action** as a homomorphism $F : \Gamma \to \text{Homeo}(X)$ (where the second group is with regards to composition). Taking $\Gamma = \mathbb{R}$, we get the definition of a flow.

We assume groups are topological groups and group actions are continuous unless otherwise stated. Taking a group action $F : \Gamma \times X \to X$ with G some group we have that a ***G*-valued**

cocycle over F is a function $\alpha : \Gamma \times X \to G$ satisfying the **cocycle identity**:

$$\alpha(\gamma_1\gamma_2, x) = \alpha(\gamma_1, F(\gamma_2, x)) \cdot \alpha(\gamma_2, x). \tag{4}$$

Problem 5. Let $X := \mathbb{T}^n = S^1 \times \cdots \times S^1$, let $\Gamma := \mathrm{Diff}(\mathbb{T}^n)$ be the group of diffeomorphisms of $\mathbb{T}^n$. Then Γ acts on X in the usual way;

$$(f, x) \mapsto f(x).$$

Recall $f \in \Gamma$ can be written locally as

$$f(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n)).$$

This gives rise to a (total) differential

$$Df(x) = \left(\frac{\partial f_i}{\partial x_j}\right)_{1 \le i,j \le n}.$$

If $G = \mathrm{GL}(n, \mathbb{R})$ under the operation of matrix multiplication and the usual Euclidean topology, show that the differential defines a cocycle $D : \Gamma \times X \to G$.

Fix an action $F : \Gamma \times X \to X$, G a group. A **coboundary** is a cocycle $\alpha : \Gamma \times X \to G$ satisfying the property that for some $\varphi : X \to G$ we have

$$\alpha(g, x) = \varphi(F(g, x))\varphi(x)^{-1}.$$

For $G = \mathbb{R}$ and $G = \mathbb{Z}$, we can rephrase this as

$$\alpha(t, x) = \varphi(F(t, x)) - \varphi(x).$$

If our group action is given by a flow (discrete or continuous) f, then we can rephrase this again as

$$\alpha(t, x) = \varphi(f^t(x)) - \varphi(x).$$

Compare this to Equation 1. If we know that $\alpha(t, x)$ is a cocycle generated by some $g : X \to \mathbb{R}$, then

$$\alpha(1, x) = g(x) = \varphi(f(x)) - \varphi(x).$$

Compare this to Theorem 1. The content of this theorem then translates to trying to determine when cocycles generated by functions are coboundaries.

Remark. We will sometimes refer to the generator as a coboundary if the cocycle it generates is a coboundary.

Claim 2. Assume now that $G = \mathbb{R}$ and we're dealing with a discrete time dynamical system $f : X \to X$. Let $g : X \to \mathbb{R}$ be a function and consider α the cocycle generated by g. There exists a solution to the cohomological equation if and only if α is a coboundary.

Proof. Recall that in this case we have α is generated by g if α is as given in Equation 3. Assume that α is a coboundary. Setting $n = 1$, we have

$$\alpha(1, x) = g(x) = \varphi(f(x)) - \varphi(x),$$

so there is a solution to the cohomological equation.

Assume now that there exists a solution to the cohomological equation. That is, there exists a φ so that

$$g(x) = \varphi(f(x)) - \varphi(x).$$

We need to show that α is a coboundary. In other words, we need to show that

$$\alpha(n, x) = \varphi(f^n(x)) - \varphi(x).$$

Notice

$$\alpha(n,x) = \sum_{i=0}^{n-1} g(f^i(x)), \quad g(f^i(x)) = \varphi(f^{i+1}(x)) - \varphi(f^i(x)),$$

so we have an alternating sum. This leaves us with the desired result for $n \geq 0$. For $n < 0$ the same kind of argument works. ☐

Before moving on, we would like to generalize the concept of a generator to flows. Ideally we would like something as in Equation 3, except we replace things with integrals instead of sums. This is almost the correct idea and we formalize it now.

Let $f^t : M \to M$ now represent a smooth flow on a smooth manifold. Let G be some Lie group with Lie algebra $\mathfrak{g}$. Given a C^1 cocycle $\alpha : \mathbb{R} \times X \to G$ the goal is to find a generator. Recall that we can find a vector field $X : M \to TM$ which generates f^t. This is given by

$$X(x) = \left.\frac{df^t(x)}{dt}\right|_{t=0}.$$

The **infinitesmal generator** $a : M \to \mathfrak{g}$ of α will be defined by

$$a(x) = \left.\frac{d\alpha(t,x)}{dt}\right|_{t=0}. \tag{5}$$

This gives us a map $\alpha(t,x) \mapsto (a : M \to \mathbb{R})$. Let's check that we can recover α from this generator. Since α a cocycle, we have the cocycle identity:

$$\alpha(s+t,x) = \alpha(t,x) + \alpha(s, f^t(x)),$$

so differentiating at $s = t$ gives us

$$\left.\frac{d\alpha(s,x)}{ds}\right|_{s=t} = \lim_{s\to 0} \frac{\alpha(s+t,x) - \alpha(t,x)}{s} = \lim_{s\to 0} \frac{\alpha(s, f^t(x))}{s}$$
$$= \lim_{s\to 0} \frac{\alpha(s, f^t(x)) - \alpha(0, f^t(x))}{s} = \left.\frac{d\alpha(s, f^t(x))}{ds}\right|_{s=0} = a(f^t(x)).$$

Consequently

$$\alpha(T,x) = \int_0^T \left.\frac{d\alpha(s,x)}{ds}\right|_{s=t} dt = \int_0^T a(f^t(x))dt.$$

So we recover α just like before.

Problem 6. Let α be a cocycle generated by a (either continuous or discrete) which is a coboundary. Show that a must satisfy the **periodic obstruction**. Namely if x is periodic, then we must have

$$\int_0^T a(f^t(x))dt = 0,$$

where we interpret this integral appropriately. Thus we see that if α is going to be a coboundary, then we must have that the periodic obstruction is satisfied.

2.2. **Flow equivalence and orbit equivalence.** We follow Section 2.2 [7] and Section 2 [8]. The definitions and results in this section will be useful in Section 4.1 and they are also useful for showing how cocycles arise in nature, but they are not necessary for understanding the proof of the Livschitz theorem.

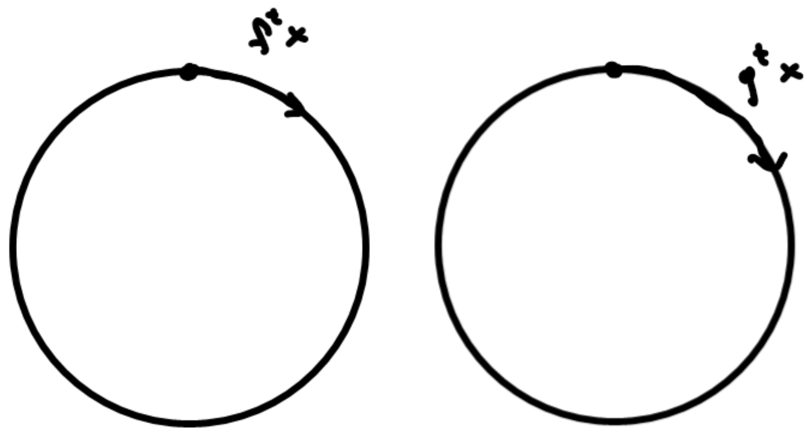

FIGURE 1. Two flows which have the same orbit but maybe don't have the same speed.

We fix two smooth flows $f^t : M \to M$, $g^t : M \to M$ on M a compact Riemannian manifolds. The flows are said to be **flow equivalent** if there exists a diffeomorphism $h : M \to M$ such that $f^t = h^{-1} \circ g^t \circ h$.

Remark. Flow equivalences can be defined for flows on different manifolds by adjusting the domain and range of h.

Note that this is an *extremely* strong statement. Unlike discrete-time flows, continuous-time flows have two structures associated to their orbits: the speed and the shape of the orbit. The above equivalence preserves both of these features, which may be too much.

A simple example is the following. Take $f^t : S^1 \to S^1$ given by $f^t(x) = x + t \pmod 1$, $g^t : S^1 \to S^1$ given by $g^t(x) = x + 2t \pmod 1$. The orbits have the same shape, however g is going much faster than f, as one can see in Figure 1. We would still like to say that such orbits are the same in some sense. This leads to the notion of a time change.

Let

$$\mathcal{O}_{f^t}(x) := \{f^t(x) : t \in \mathbb{R}\}$$

denote the orbit of a point $x \in M$ under the flow f^t. We say f^t is a **time change** of g^t if for each $x \in M$ we have

$$\mathcal{O}_{f^t}(x) = \mathcal{O}_{g^t}(x)$$

and the orientations given by the change of t in the positive direction are the same.

Claim 3. If g^t is a time change of f^t, then $g^t(x) = f^{\alpha(t,x)}(x)$ for every $x \in M$, where $\alpha : \mathbb{R} \times M \to \mathbb{R}$ is a cocycle over g satisfying $\alpha(t, x) \geq 0$ for $t \geq 0$.

Sketch of proof. We show α is a cocycle. The orbits are the same for every point, so fixing x we can define a function $\alpha(t, x) : \mathbb{R} \to \mathbb{R}$ where $f^{\alpha(t,x)}(x) = g^t(x)$. This function is well-defined by construction. It remains to check that it satisfies the cocycle identity. We check the following:

$$f^{\alpha(t,g^s(x))+\alpha(s,x)}(x) = f^{\alpha(t,g^s(x))}(f^{\alpha(s,x)}(x)) = g^t(g^s(x)) = g^{t+s}(x) = f^{\alpha(t+s,x)}(x).$$

Here we should be careful, since x may be periodic and therefore we might not have the functions are actually equal. As long as we restrict things to the correct domains, it turns out this is a non-issue. Thus we see

$$\alpha(t + s, x) = \alpha(t, g^s(x)) + \alpha(s, x),$$

which is precisely what we needed. □

Problem 7. Finish the proof of Claim 3 by showing $\alpha(t,x) \geq 0$ for $t \geq 0$. Moreover, show that if x is not a periodic, $\alpha(t,x) > 0$ for $t > 0$.

The question we wish to study now is the following: When does a time change produce a flow that is flow equivalent to the original? Suppose g is a time change of f, so $g^t(x) = f^{\alpha(t,x)}(x)$. Then we can boil the question down to whether we can find $h : M \to M$ so that

$$f^t(x) = h \circ g^t \circ h^{-1}(x).$$

A simple case occurs when

$$h(x) = f^{\beta(x)}(x),$$

where $\beta : X \to \mathbb{R}$ is some differentiable function with derivative in the direction of the flow f;

$$\left.\frac{d\beta(f^t(x))}{dt}\right|_{t=0} = \lim_{t\to 0} \frac{\beta(f^t(x)) - \beta(x)}{t}.$$

We have two conditions which arise from this β:

$$f^t(h(x)) = f^t(f^{\beta(x)}(x)) = f^{t+\beta(x)}(x), \tag{6}$$

and

$$h \circ g^t(x) = f^{\beta(g^t(x))}(f^{\alpha(t,x)}(x)) = f^{\beta(g^t(x))+\alpha(t,x)}(x). \tag{7}$$

The above equations must be equal, so rewriting gives us

$$\alpha(t,x) + \beta(g^t(x)) = t + \beta(x)$$

or

$$t - \alpha(t,x) = \beta(g^t(x)) - \beta(x). \tag{8}$$

A time change satisfying Equation 8 is said to be **trivial**. Note the correspondence between this and coboundaries.

Problem 8. Show in the above scenario we have $t - \alpha(t,x)$ is a cocycle (over g) by showing the cocycle identity. Conclude that a time change produces a flow that is flow equivalent to the original if we have this cocycle is a coboundary.

Observe that

$$\lim_{t\to 0} \frac{\alpha(t,h(x)) - t}{t} = \lim_{t\to 0} \frac{\beta(f^t(x)) - \beta(x)}{t} = \left.\frac{d\beta(f^t(x))}{dt}\right|_{t=0}.$$

If α has generator a, then

$$\lim_{t\to 0} \frac{\alpha(t,h(x)) - \alpha(0,h(x))}{t} - 1 = a(h(x)) - 1 = \left.\frac{d\beta(f^t(x))}{dt}\right|_{t=0}.$$

We will return to these ideas in Section 4.1.

2.3. **Hyperbolicity.** We follow Chapter 6 [7] and Section 4.6 [17].

In order to prove Theorem 1 we will need to use density of periodic orbits and the vanishing condition in order to find a solution to the equation. This section focuses on introducing these topics.

Here, $f : M \to M$ denotes a smooth diffeomorphism of a compact Riemannian manifold. Recall a set $J \subseteq M$ is **invariant** if $f^t(J) \subseteq J$ for all $t \in \mathbb{Z}$. An invariant set $\Lambda \subseteq M$ is **hyperbolic** if for each $x \in \Lambda$ we have the splitting $T_xM = E^s(x) \oplus E^u(x)$ which is invariant as a family in the sense that

$$Df(E^s(x)) = E^s(f(x)), \quad Df(E^u(x)) = E^u(f(x)),$$

and we have that there exists $C \geq 1$ and $0 < \lambda < 1$ so that the following holds:

$$|Df^n(v)| \leq C\lambda^n |v|, \quad \forall x \in \Lambda, v \in E^s(x), n \geq 0,$$
$$|Df^{-n}(v)| \leq C\lambda^n |v|, \quad \forall x \in \Lambda, v \in E^u(x), n \geq 0.$$

Problem 9. Show that Λ being hyperbolic is independent of the metric on M.

A nice feature of hyperbolic systems is that we can view things locally. We discuss this feature now. First, we have the following extension result.

Lemma (Extension Lemma, Lemma 6.2.7 [7])**.** Let U be an open bounded neighborhood of $0 \in \mathbb{R}^n$. Let $f : U \to \mathbb{R}^n$ be a diffeomorphism which fixes the origin. For $\epsilon > 0$ there exists $\delta > 0$ and a diffeo $\widehat{f} : \mathbb{R}^n \to \mathbb{R}^n$ such that $\|\widehat{f} - Df_0\|_{C^1} < \epsilon$ and $\widehat{f} = f$ on $B(0, \delta)$.

Proof. Fix $\epsilon > 0$. Let $\epsilon > \eta > \delta > 0$, where we will choose η and δ later. We can find a C^1 bump function $\rho : \mathbb{R}^n \to [0, 1]$ such that $\rho = 1$ on $B(0, \delta)$ and $\rho = 0$ off $B(0, \eta)$. We can also choose it so that $\|D\rho\| \leq C_0/\eta$. Set

$$\widehat{f} := \rho \cdot f + (1 - \rho)Df_0,$$

where we have $\rho \cdot f = 0$ when $\rho = 0$, even if f is undefined. Then $\widehat{f} - Df_0 = \rho(f - Df_0)$. Since f is C^1, we get that $\|f - Df_0\|_{C^0} = o(\eta)$ and

$$\|D(\widehat{f} - Df_0)\| \leq \|D\rho(f - Df_0)\| + \|\rho(Df - Df_0)\| = o(1) \text{ in } \eta.$$

Shrink η and choose appropriate δ so that the result holds. □

The above result is useful in defining a localization procedure, which we now describe. This procedure can be found in Section 6.4 (b) [7]. For each $x \in \Lambda$, we can fix a coordinate system in the tangent space T_xM such that $E^+_x \oplus E^-_x$ is identified with $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^{n-k}$. The Riemannian metric in T_xM in this case becomes the standard Euclidean metric. Fix a small enough $\epsilon > 0$ so that the exponential map $\exp_x : D_\epsilon \to M$ is a diffeomorphism onto its image. Thanks to the compactness of Λ, we can choose a uniform ϵ for all of Λ. By conjugating, we get a family of map

$$f_{x,\epsilon} = \exp^{-1}_{f(x)} \circ f \circ \exp_x : D_\epsilon \to \mathbb{R}^n.$$

Since we're using the exponential map, we have that for sufficiently small ϵ the map $f_{x,\epsilon}$ is C^1-close to its differential $(Df)_x$ expressed in local coordinates. We then use the extension lemma to get maps $f_{x,\epsilon}$ on all of $\mathbb{R}^n$. Relabel these as f_x, then along each orbit $\{f^m(x)\}_{m\in\mathbb{Z}}$ we get a sequence of maps $f_m = f_{f^m(x)} : \mathbb{R}^n \to \mathbb{R}^n$, and these maps look like

$$f_m(x, y) = (A_m(x) + \alpha_m(x, y), B_m(y) + \beta_m(x, y))$$

with the C^1-norm of α, β depending on ϵ. By shrinking ϵ even more we can make these norms as small as we want. We refer to the above as the **localization procedure**, and it allows us to study the dynamics of f locally in these coordinates.

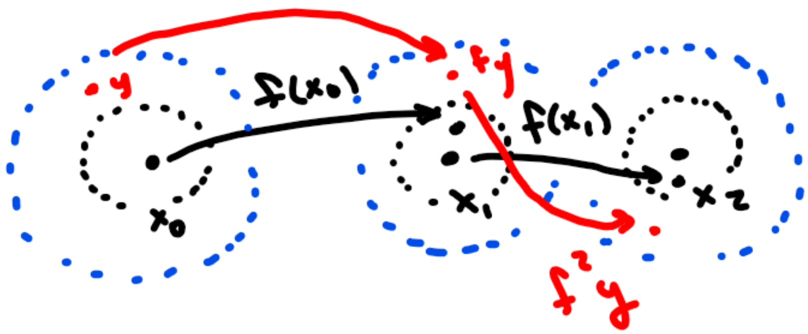

FIGURE 2. A picture of shadowing. The black denotes δ balls, the blue denotes ϵ balls.

We say a sequence $(x_n)_{n=-\infty}^{\infty}$ in M is a **δ-pseudo-orbit** of f if for every $n \in \mathbb{Z}$ we have

$$d(f(x_n), x_{n+1}) < \delta.$$

We say a point $y \in M$ **ϵ-shadows** a pseudo-orbit $(x_n)_{n=-\infty}^{\infty}$ if for every $n \in \mathbb{Z}$ we have

$$d(f^n(y), x_n) < \epsilon.$$

Figure 2 gives a picture of what's going on.

The following lemma is critical for the proof of Theorem 1.

Theorem 2 (Anosov closing lemma, Theorem 6.4.15 [7])**.** Let Λ be a hyperbolic set for $f : U \to M$. Then there exists an open neighborhood V with $\Lambda \subseteq V$ and $C, \epsilon_0 > 0$ such that for $\epsilon < \epsilon_0$ and any periodic ϵ-orbit $(x_0, \dots, x_m) \subseteq V$ there is a point $y \in U$ such that $f^m(y) = y$ and $d(f^k(y), x_k) < C\epsilon$ for $0 \leq k \leq m-1$.

Proof of Theorem 2. Apply the localization procedure. For each $x \in \Lambda$ this produces a neighborhood V_x so that we can reformulate things in terms of small perturbations of hyperbolic linear maps. Label the maps along the orbit as

$$f_k(u, v) = (A_k(u) + \alpha_k(u, v), B_k(v) + \beta_k(u, v))$$

with $\|\alpha_k\|_{C^1}, \|\beta_k\|_{C^1} < C_1\epsilon$ for some $C_1 > 0$. Now consider the map

$$F : (\mathbb{R}^k \times \mathbb{R}^{n-k})^m \to (\mathbb{R}^k \times \mathbb{R}^{n-k})^m,$$

$$F((u_0, v_0), \dots, (u_{m-1}, v_{m-1})) := (f_{m-1}(u_{m-1}, v_{m-1}), f_0(u_0, v_0), \dots, f_{m-2}(u_{m-2}, v_{m-2})).$$

We can view this map as taking a sequence of points and moving them to their corresponding next point according to the maps f. A visualization of F is given in Figure 3.

A sequence of such points will be periodic if it is a fixed point of this map. The goal now is to use the contraction mapping principle, so we'll need to choose an appropriate norm to make this map contracting. We equip $(\mathbb{R}^k \times \mathbb{R}^{n-k})^m$ with the max norm,

$$\|(x_0, \dots, x_{m-1})\| = \max_{0 \leq i \leq m-1} \|x_i\|.$$

Rewrite the coordinates as

$$(u, v) = ((u_0, v_0), \dots, (u_{m-1}, v_{m-1})).$$

In these coordinates we can express F as the sum of two maps

$$F(u, v) = L(u, v) + S(u, v),$$

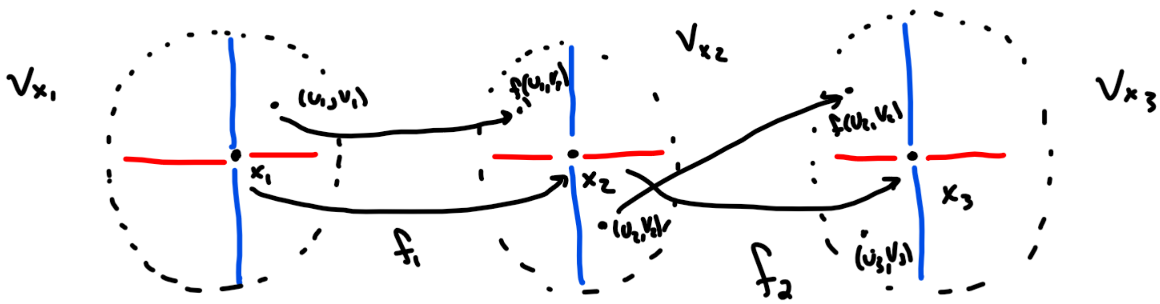

FIGURE 3. A visualization of what F is doing to each point in three different coordinate neighborhoods.

where

$$L(u,v) := ((Am-1(u_{m-1}), B_{m-1}(v_{m-1}), \dots (A_{m-2}(u_{m-2}), B_{m-2}(v_{m-2})))$$

and

$$S(u,v) := ((\alpha_{m-1}(u_{m-1}, v_{m-1}), \beta_{m-1}(u_{m-1}, v_{m-1})), \dots, (\alpha_{m-2}(u_{m-2}, v_{m-2}), \beta_{m-2}(u_{m-2}, v_{m-2}))).$$

We note that L is hyperbolic, so $(L - \mathrm{Id})$ is invertible and $\|(L - \mathrm{Id})^{-1}\| \leq C_2$ for some $C_2 > 0$. We also get $\|S(u,v) - S(u',v')\| \leq C_3 \cdot \epsilon \cdot \|(u,v) - (u',v')\|$ with $C_3 > 0$. Now observe

$$F(z) = z \iff L(z) + S(z) = z \iff S(z) = -(L - \mathrm{Id})(z) \iff -(L - \mathrm{Id})^{-1} S(z) = z.$$

We define a map

$$\mathcal{F} : (\mathbb{R}^k \times \mathbb{R}^{n-k})^m \to (\mathbb{R}^k \times \mathbb{R}^{n-k})^m,$$

$$\mathcal{F}(z) = -(L - \mathrm{Id})^{-1} S(z).$$

If we take $\epsilon < 1/C_2C_3$, then we notice

$$\|\mathcal{F}(z) - \mathcal{F}(z')\| \leq C_2 C_3 \epsilon \|z - z'\| < \|z - z'\|,$$

so $\mathcal{F}$ is a contraction. Applying the contraction mapping principle gives a unique fixed point for this map, and thus we get a unique solution to F.

Now notice in our localization procedure we've extended things to all of $\mathbb{R}^n$. The issue with this is that the fixed point we've constructed may have some point which leaves the neighborhood V_{x_k}. This would mean that our periodic point does not correspond to an actual periodic point in our original manifold. We claim this doesn't happen. That is, we claim the distance to x_k does not exceed $C\epsilon$ for some $C > 0$. Write $x = (x_0, \dots, x_{m-1})$, $z_0 = \lim_{i\to\infty} \mathcal{F}^i(x)$. Then

$$\|z_0 - x\| \leq \sum_{i=1}^{\infty} \|\mathcal{F}^i(x) - \mathcal{F}^{i-1}(x)\|.$$

Observe

$$\|\mathcal{F}^k(x) - \mathcal{F}^{k-1}(x)\| \leq C_2 C_3 \epsilon \|\mathcal{F}^{k-1}(x) - \mathcal{F}^{k-2}(x)\| \leq (C_2 C_3 \epsilon)^{k-1} \|\mathcal{F}(x) - x\|.$$

So

$$\|z_0 - x\| \leq \left(\sum_{k=0}^{\infty} (C_2 C_3 \epsilon)^k \right) \|\mathcal{F}(x) - x\|.$$

Now observing $Fx = x + v$ for some v with $\|v\| < \epsilon$ (the points in x represent the origin, so the distance after an iteration is only getting perturbed by the α_m, β_m which have the bound ϵ) we have

$$\mathcal{F}(x) = x - (L - \mathrm{Id})^{-1} v \implies \|\mathcal{F}(x) - x\| \leq C_2 \epsilon.$$

This finishes the proof. □

We can actually strengthen it with the following results.

Proposition 1 (Improved closing lemma, Proposition 6.4.16, Corollary 6.4.17 [7])**.** Let Λ be a hyperbolic set for $f : U \to M$ with λ as in the definition of a hyperbolic set. Then for $\alpha > \lambda$, there exists a neighborhood V with $\Lambda \subseteq V$ and $C_1, \epsilon_0 > 0$ so that if $f^k(x) \in V$ for $0 \leq k \leq n$ and $d(f^k(x), x) < \epsilon_0$ then there is a periodic point y so that $f^n(y) = y$ and

$$d(f^k(y), f^k(x)) < C_1 \alpha^{\min(k, n-k)} d(f^n(x), x).$$

This strengthening is not too hard once we have the closing lemma, however we would need to some more results on hyperbolic sets in order to prove it.

There is also a closing lemma for flows which is important for the flow analogue of Theorem 1.

Theorem 3 (Improved closing lemma for flows, Lemma 1 [8])**.** Let M be a Riemannian manifold and f^t a C^1 flow, $\Lambda \subseteq M$ a compact locally maximal hyperbolic set. Then for all large enough $\alpha \in (0, 1)$ there is an open neighborhood V of Λ and constants $C, \delta > 0$ so that if $x \in \Lambda$ satisfies $d(f^s(x), x) < \delta$ then there is a periodic point $y \in \Lambda$ with period T satisfies $|T - s| \leq C\delta$ so that for $0 \leq t \leq s$ we have

$$d(f^t(x) f^t(y)) \leq C \alpha^{\min(t, s-t)} d(f^s(x), x)$$

Comparing this to Proposition 1, we see these are almost identical.

We finish by discussing briefly specification, which will be useful for generalizations of Theorem 1. Let (X, f) be a discrete time dynamical system. A **specification** $S = (\tau, P)$ is a collection of finite intervals

$$\tau = \{I_1, \ldots, I_m\}, I_j = [a_j, b_j] \subseteq \mathbb{Z},$$

and a map

$$P : \bigcup_{j=1}^{m} I_j \to X$$

satisfying the property that if $t_1, t_2 \in I_j \in \tau$, then

$$f^{t_2 - t_1}(P(t_1)) = P(t_2).$$

Intuitively, we are just taking m different orbit segments and thinking of this as a set.

We say a specification S is **ϵ-shadowed** by x if

$$d(f^n(x), f^n(y)) < \epsilon \text{ for all } n \in I.$$

A specification is **M-spaced** if $a_{i+1} > b_i + M$.

We say a system (X, f) has the **specification property** if for every $\epsilon > 0$ there is an $M \in \mathbb{N}$ such that every M-spaced specification is ϵ-shadowed by some $x \in X$, and furthermore for any $q \geq M + (b_m - a_1)$, there is a period q orbit ϵ-shadowing the specification. A picture is given in Figure 4.

For flows, we take the above definitions and simply replace $\mathbb{Z}$ with $\mathbb{R}$.

Theorem 4 (Theorem 18.3.14 [7])**.** Let Λ be a topologically mixing compact locally maximal hyperbolic set for a smooth flow $f^t : M \to M$. Then $f^t|_\Lambda$ has the specification property.

Corollary. Geodesic flows have the specification property.

FIGURE 4. A picture of the specification property. The black segments denote the orbit segments, the blue segment denotes the orbit shadowing.

3. Proof and extensions

3.1. Proof of the Livschitz theorem. We prove Theorem 1 following Section 19.2 [7].

Let's try approaching the problem blindly. We have a cohomological equation we wish to solve, say

$$g = \varphi \circ f - \varphi.$$

Since we are on a topologically transitive set, let's take $x \in \Lambda$ with a dense orbit and try setting

$$\varphi(f^n(x)) = \sum_{i=0}^{n-1} g(f^i(x)).$$

Since the orbit of this point is dense, we have this captures almost every point (in a topological sense) and on the orbit it satisfies the above cohomological equation. We've now defined φ on a dense set, and we would like to extend it from this dense set to the entire space. The first issue is that there might be some hidden data which changes things drastically, ruining this approach. The second issue comes from periodic obstruction – it's not enough to define it on above, we need to know apriori that it vanishes on periodic orbits. Using our work in Section 2.3 it may seem clear that we can solve both issues at once. By assumption, we know that we have vanishing on periodic orbits, and by hyperbolicity we know these periodic orbits approximate every orbit. Since g is Hölder, it seems hopeful that this should be enough to finish the theorem. We proceed with the proof now.

Proof of Theorem 1. We do exactly as described above. Choose a point $x_0 \in \Lambda$ so that $\mathcal{O}_{f^n}(x_0) \subseteq \Lambda$ is dense. It's clear from the discussion prior to the proof we just need to know the value of φ at the point x_0 to see how it's defined everywhere else. Thus we choose some value $y \in \mathbb{R}$ and we set $\varphi(x_0) = y$. Now we do exactly as we predicted. Define

$$\varphi(f^n(x_0)) = \varphi(x_0) + \alpha(n, x_0),$$

where α is a cocycle generated by g, as defined in Equation 3. To finish, we break the proof up into three steps.

Step 1: We show φ is Hölder continuous on $\mathcal{O}_{f^n}(x_0)$ with the same exponent as g. Choose $n, m \in \mathbb{Z}$ so that $\epsilon := d(f^n(x_0), f^m(x_0))$ is small enough to apply Proposition 1. We can then find $C > 0$, $0 < \mu < 1$, and $y = f^{m-n}(y)$ so that

$$d(f^{n+i}(x_0), f^i(y)) \leq C\epsilon\mu^{\min(i, m-n-i)}.$$

Since g is Hölder continuous, there exists $M > 0$ so that whenever $d(x_1, x_2)$ is small we have

$$|g(x_1) - g(x_2)| \leq M d(x_1, x_2)^{\alpha}.$$

Now we use the vanishing of periodic orbits and the triangle inequality to get

$$\begin{aligned}|\varphi(f^n(x_0)) - \varphi(f^m(x_0))| &= \left|\sum_{i=0}^{m-n-1} g(f^{n+i}(x_0))\right| \\ &= \left|\sum_{i=0}^{m-n-1} [g(f^{n+i}(x_0)) - g(f^i(y))]\right| \\ &\leq \sum_{i=0}^{m-n-1} |g(f^{n+i}(x_0)) - g(f^i(y))|.\end{aligned}$$

Since g is Hölder, we have

$$\begin{aligned}\sum_{i=0}^{m-n-1} |g(f^{n+i}(x_0)) - g(f^i(y))| \leq MC^\alpha\epsilon^\alpha \sum_{i=0}^{m-n-1} \mu^{\alpha \min(i,m-n-i)} \\ < MC^\alpha\epsilon^\alpha \sum_{i=0}^{\infty} \mu^{\alpha i} \\ = MC^\alpha\epsilon^\alpha \frac{1}{1-\mu^\alpha} \\ = \frac{MC^\alpha}{1-\mu^\alpha} d(f^n(x_0), f^m(x_0))^\alpha.\end{aligned}$$

This shows that φ is Hölder as well, hence uniformly continuous on the orbit.

Step 2: Being uniformly continuous on a dense set means we can extend it to a continuous function on all of Λ. Moreover, this extension shares the same Hölder exponent. Now $g = \varphi \circ f - \varphi$ on a dense set, and since they are continuous when we extend they are still equal. This establishes existence.

Step 3: The difference of two solutions is a continuous f-invariant function, so it must be constant.

□

Problem 10. Make Step 3 more formal by showing the following.

(1) Let $f : X \to X$ be a continuous map of a compact metric space. Show f is topologically transitive if and only if for any two nonempty open sets $U, V \subseteq X$ there is an integer N so that $f^N(U) \cap V \neq \varnothing$.
(2) Show that $f : X \to X$ above is topologically transitive if and only if there are no two disjoint open nonempty f-invariant sets.
(3) Show that if $f : X \to X$ above is topologically transitive then there is no f-invariant nonconstant continuous function $\varphi : X \to \mathbb{R}$.

Problem 11. If there exists a continuous $\varphi : \Lambda \to \mathbb{R}$ such that

$$g = \varphi \circ f - \varphi$$

then can we say that for every $x \in M$ with period n we have

$$\sum_{i=0}^{n-1} g(f^i(x)) = 0?$$

Notice that this is would give us a backwards direction to Theorem 1 – coboundaries must vanish on all periodic orbits.

3.2. **Extensions of the Livschitz theorem.** We now give a few extensions of Theorem 1 following Section 19.2 [7] and Section 6.3 [3].

We note that Theorem 1 follows if we impose more structure on g. The interesting part is that, in some cases, imposing more structure on g gives us that same structure on φ. We improve the theorem as follows.

Theorem 5. Let M be a Riemannian manifold, $f : U \to M$ a smooth embedding with a compact topologically transitive hyperbolic set Λ and $g : \Lambda \to \mathbb{R}$ is either Hölder continuous or C^1. Suppose that for every $x \in M$ with period n we have

$$\sum_{i=0}^{n-1} g(f^i(x)) = 0.$$

Then there exists a continuous $\varphi : \Lambda \to \mathbb{R}$ such that

$$g = \varphi \circ f - \varphi.$$

Moreover φ is unique up to an additive constant, and either Hölder with the same exponent as g or C^1 depending on g.

To prove this, we need an additional result.

Claim 4. If $f : \mathbb{R}^n \to \mathbb{R}$ is C^1 along the leaves of two continuous transverse foliations, then f is C^1.

Proof. Let A, B be the two foliations. The goal is to show that for y near x there is a linear map L_x satisfying

$$f(y) - f(x) = L_x(x - y)$$

up to higher-order terms in $|x-y|$. Let $z \in A(x) \cap B(y)$. We know that f is C^1 along the foliations, so

$$f(x) - f(z) = L_z^A(x - z),$$

and similarly

$$f(y) - f(z) = L_z^B(y - z),$$

with both L_z^A and L_z^B depending continuous on the basepoint z. Adding and subtracting $f(z)$ and using the above grants us

$$f(x) - f(y) = [f(x) - f(z)] + [f(z) - f(y)] = L_z^A(x - z) + L_z^B(z - y),$$

up to higher order terms. Now as $z \to x$, we get $L_z^B \to L_x^B$. So taking $y \to x$, i.e. up to higher order terms, we get

$$L_z^B(y - z) = L_x^B(y - z).$$

We can define L_x to be the linear map which restricts to L^A on $TA(x)$ and L^B on $TB(x)$, and this gives the desired result. □

Proof of Theorem 5. Suppose g is C^1. By the above argument, we get that it is Lipschitz, so differentiable almost everywhere. If x and y are nearby points on a stable leaf, then

$$\begin{aligned}\varphi(y) - \varphi(x) &= \lim_{n\to\infty} \left(-\sum_{i=0}^{n} (g(f^i(y)) - g(f^i(x)) + g(f^n(x)) - g(f^n(y)) \right) \\ &= \sum_{i=0}^{\infty} (g(f^i(y)) - g(f^i(x)).\end{aligned}$$

Keeping x fixed, let $y = x + tv$ and differentiate at $t = 0$. This grants us

$$D_v\varphi(x) = -\sum_{i=0}^{\infty} D_{v_i}(g(f^i(x))D_v(f^i)(x),$$

where $v_i := Df^i v$. Note the series converges uniformly since everything is on a stable leaf and g is C^1. The unstable direction is the same. This shows that φ has C^1 derivatives along the stable and unstable leaves, so using Claim 4 we get our result. □

Remark. The same result can be shown if one replaces C^1 with C^∞.

It turns out the same result holds for flows.

Theorem 6. Let M be a Riemannian manifold, $\Lambda \subseteq M$ a compact locally maximal hyperbolic set for a flow f^t. Let $g : \Lambda \to \mathbb{R}$ a Hölder continuous function such that if $\varphi^T(x) = x$ then

$$\int_0^T g(f^t(x))dt = 0.$$

If there is $x \in \Lambda$ such that $\{f^t(x) : t \in \mathbb{R}\} \subseteq \Lambda$ is dense, then there is a continuous $\varphi : \Lambda \to \mathbb{R}$ such that

$$g(y) = \left.\frac{d\varphi(f^t(y))}{dt}\right|_{t=0}.$$

Moreover φ is unique up to an additive constant, and either Hölder with the same exponent as g or C^1 depending on g.

Sketch of proof. The idea is the same as the proof of Theorem 1. Define

$$\varphi(f^t(x)) := \int_0^t g(f^s(x))ds.$$

Then one needs to show that φ is uniformly continuous, hence has a unique continuous extension to all of Λ by density. In showing it is uniformly continuous, we also show that it is Hölder continuous with the same exponent as g. To show that it is Hölder continuous, we use Theorem 3 and the vanishing property. The argument for uniqueness is similar to Theorem 1. Finally, the C^1 argument follows exactly the same as the prior theorem. □

Problem 12. Formalize the prior proof by working out the details.

Remark. The same result holds if we replace C^1 with C^∞. This is called the cocycle regularity theorem; see [6] and [11].

Let M be a Riemannian manifold, $\Lambda \subseteq M$ a compact locally maximal hyperbolic set for a flow f^t, where this flow is possibly with discrete time. Let $g : \Lambda \to \mathbb{R}$ be a Hölder continuous function. Let

$$\text{Per}(f^t) := \{x \in \Lambda : f^T(x) = x \text{ for some } T > 0\}.$$

We say that g satisfies the **vanishing condition** on some subset $Q \subseteq \text{Per}(f^t)$ if for every $x \in Q$ with $f^T(x) = x$ for some $T > 0$ we have

$$\int_0^T g(f^t(x))dt = 0.$$

Notice the Livschitz theorem above holds when g satisfies the vanishing condition for $Q = \text{Per}(f^t)$.

The goal for the remainder of this section is to try and figure out whether we are able to relax the condition that we need to check g vanishes on all periodic orbits. To make this idea more formal, we phrase the question as follows: Does Livschitz hold if we have this vanishing condition on some large subset $Q \subseteq \mathrm{Per}(f^t)$? One clear observation is that if it holds on Q, then it must hold on all periodic orbits, so it needs to be a set which approximates periodic orbits well. We give an example to show that being dense is not enough.

Example. Consider the left-shift map on two symbols:

$$\Omega_2^R := \{(x_i)_{i=0}^{\infty} : x_i \in \{0,1\}\},$$
$$\sigma : \Omega_2^R \to \Omega_2^R, \quad \sigma((x_i)) = (y_i),\ y_i = x_{i+1}.$$

Define

$$g : \Omega_2^R \to [0,\infty), \quad g((x_i)) = (-1)^{x_0}.$$

Observe that the set where g vanishes on is the set of periodic binary sequences with the same number of 0s and 1s. Observe this set is dense in Λ, since taking any cylinder we can add on as many 0s and 1s as necessary to get a sequence with the same number of 0s and 1s. On the other hand, g doesn't satisfy the conditions for the Livschitz theorem to hold. The key observation to make here is that asymptotically the number of points where g vanishes on grows exponentially small as we let the length of the sequence go to infinity.

Define the metric on Ω_2^R to be

$$d((x_i),(y_i)) = 2^{-k}, \text{ where } x_i = y_i \text{ for } 0 \le i \le k, x_{k+1} \neq y_{k+1}.$$

Let σ be the left-shift map as above. Define

$$P(n) := \{x \in \Omega_2^R : \sigma^n(x) = x\}.$$

Let

$$Q := \left\{ x \in \mathrm{Per}(\sigma) : \text{ if } x \text{ has period N, then } \sum_{i=0}^{N-1} g(\sigma^i(x)) = 0 \right\}.$$

Define $Q(n) := Q \cap P(n)$. We say that $Q(n) \subseteq P(n)$ is $\epsilon(n)$**-dense** if for all $x \in P(n)$, there is a $y \in Q(n)$ with $d(x,y) < \epsilon(n)$.

Proposition 2. Let $\sigma : \Omega_2^R \to \Omega_2^R$ be the left-shift map. If $Q(n)$ is $\epsilon(n)$-dense, where $\epsilon(n) = O(2^{-n})$[5], then $g : \Omega_2^R \to \mathbb{R}$ Hölder continuous satisfies the vanishing on Q implies that g satisfies the vanishing condition on $\mathrm{Per}(\sigma)$.

Proof. Take $x \in P(n)$. Observe that $x \in \bigcap_{k\ge1} P(kn)$. Let y_k be such that $y_k \in Q(kn)$ and $d(p, y_k) < \epsilon(kn)$ for some $p = \sigma^i(x)$ with $i \ge 0$. Then

$$\begin{aligned}
\left|\sum_{i=0}^{n-1} g(\sigma^i(x))\right| &= \frac{1}{k}\left|\sum_{i=0}^{kn-1} g(\sigma^i(p))\right| \\
&= \frac{1}{k}\left|\sum_{i=0}^{kn-1} [g(\sigma^i(p)) - g(\sigma^i(y_k))]\right| \\
&\le \frac{C}{k}\sum_{i=0}^{kn-1} d(\sigma^i(p), \sigma^i(y_k))^{\alpha} \\
&\le \frac{C}{k} d(p, y_k) \sum_{i=0}^{kn-1} 2^{i\alpha} \\
&< \frac{C}{k}\epsilon(kn)\left(\frac{1-2^{kn\alpha}}{1-2^{\alpha}}\right).
\end{aligned}$$

[5]Recall $f(n) = O(g(n))$ if $|f(n)| \le Mg(n)$ for some $M > 0$ and for all n sufficiently large.

We can do this for all $k \geq 1$, and since the choice of k is arbitrary in our construction we can let k tend to infinity to get the sum is 0. □

Remark. Observe that if we take an equivalent metric on Ω_2^R then the same proof holds, since we change things by at most a constant.

Remark. This shows that both the number *and* the location matters in our choice of Q. Can we relax it to just number? That is, if we know that $|Q(n)|/|P(n)| \geq C$ for all n, then can we say that Livschitz still holds?

Claim 5. Let $\sigma : \Omega_2^R \to \Omega_2^R$ be the left-shift map. Let $g : \Omega_2^R \to \mathbb{R}$ be Hölder continuous and satisfying the vanishing condition on Q with $|Q(n)|/|P(n)| \geq C$. If $x \in \Omega_2^R$ is periodic with period m and not necessarily in $Q(m)$, then for

$$n = -\frac{\ln(C)}{\ln(2) - \ln(2 - 2^{-m})} + 1$$

we have that adding on $n - m$ symbols gives us a point in $Q(n)$.

Proof. Suppose we have an periodic point with period m, so $(x_i)_{i=0}^{\infty}$ satisfying $x_i = x_{i+m}$ for every i. Create a cylinder

$$\mathcal{C} = \{(y_i) : y_i = x_i \text{ for } 0 \leq i \leq m-1\}.$$

Consider

$$\Gamma(n, m) = |\{x \in P(n) : \mathcal{O}(x) \cap \mathcal{C} = \varnothing\}|.$$

This is counting the number of points whose orbits don't land in the cylinder. Notice the worst case scenario is when we have a constant periodic sequence (this follows by an easy counting argument), so we study this particular case to try to figure out how many symbols we need to add on. Create a new alphabet with $2^m - 1$ symbols consisting of all binary sequences of length m except for the sequence with all 0s. Define a graph with nodes from this alphabet, and connect two nodes if the last $m-1$ letters match the first $m-1$ letters (so this is directed graph). It is an easy argument to see that this graph is connected. Observe this graph being connected implies that the corresponding connectivity matrix is transitive. We then use Corollary 1.9.5 [7] coupled with Perron-Frobenius (Theorem 1.9.11 and Corollary 1.9.12 [7]) to get that the trace of the powers of this matrix is bounded by the largest eigenvalue to the same power. Using Newton's method and some facts on the characteristic polynomial, we get a bound for the largest eigenvalue:

$$\lambda_{\max} \leq 2 - 2^{-m} < 2.$$

Furthermore, it follows that this gives us a bound for $\Gamma(n, m)$:

$$\Gamma(n, m) \leq (2 - 2^{-m})^n.$$

Now observe that if $\Gamma(n, m)/2^n < C$, we get that adding on $n - m$ symbols gives us something in $Q(n)$ ($\Gamma(n, m)/2^n < C$ says that the proportion of points whose orbits miss the cylinder entirely is less than C, and since $|Q(n)|/|P(n)| > C$ there must be some point in $Q(n)$ which hits our cylinder). Using our bounds, we just need to figure out for fixed m what n gives us

$$\frac{(2 - 2^{-m})^n}{2^n} < C.$$

Solving gives us the above result. □

Remark. It seems as though the bounds in the above argument are tight, which is a bad thing for our argument in Proposition 2. Ideally, we'd like to use specification to approximate our orbit, and then in some sense use specification again to approximate the new orbit with one that vanishes. Since the above is not a bounded number, we can't simply divide by k to fix things.

We now move onto flows. First, we give an example for geodesic flows.

Proposition 3. Let W be a compact Riemannian manifold, $f^t : T^1W \to T^1W$ the geodesic flow. Let $g : T^1W \to \mathbb{R}$ be Hölder continuous and notice for every $x \in \text{Per}(f^t)$ with period T we have $y \in \text{Per}(f^t)$ with period T which gives the backwards orbit. Suppose one of the following happens:

$$\int_0^T g(f^t(x))dt = 0 \text{ or } \int_0^T g(f^t(y))dt = 0.$$

Then g satisfies the vanishing condition on all of $\text{Per}(f^t)$.

Proof. Take $x \in \text{Per}(f^t)$ with period T, denote the backward orbit with $y \in \text{Per}(f^t)$. Fix a positive integer k, and consider $I = [0, kT]$. Let $s' = kT + M$, M to be determined. By specification, we know that for some M chosen appropriately we can find z such that z is periodic with period s satisfying $|s - s'| < \epsilon$, and which satisfies

$$d(f^t(x), f^t(z)) < \epsilon \text{ for } t \in I.$$

Without loss of generality, we may assume that

$$\int_0^s g(f^t(z))dt = 0,$$

since if not we can take its opposite and its opposite still satisfies the above property. Then we can write

$$\left|\int_0^T g(f^t(y))dt\right| = \frac{1}{k}\left|\int_0^{kT} g(f^t(x))dt - \int_0^s g(f^t(z))dt\right|$$
$$= \frac{1}{k}\left|\int_0^{kT} [g(f^t(x)) - g(f^t(z))] + \int_{kT}^s g(f^t(z))dt\right|.$$

Using the fact that g is Hölder, we can rewrite the above as

$$\left|\int_0^T g(f^t(y))dt\right| \le \frac{1}{k}\left[C\epsilon^\alpha kT + (M+\epsilon)\sup_t |g(f^t(z))|\right].$$

Observe two things. First, since W is compact and g is Hölder continuous, we get that g is uniformly bounded by some constant N, so we get an upper bound

$$\left|\int_0^T g(f^t(y))dt\right| < \frac{1}{k}\left[C\epsilon^\alpha kT + (M+\epsilon)N\right].$$

Next, observe that M and N do not depend on k. The above inequality holds for all k, so letting k tend to infinity we get

$$\left|\int_0^T g(f^t(y))dt\right| < C\epsilon^\alpha T.$$

Finally this holds for all $\epsilon > 0$, so let ϵ tend to zero to get the desired result. ☐

We could also mimic Proposition 2. We say $Q(T)$ is $\epsilon(T)$**-dense** in $P(T)$ if for every $\gamma \in P(T)$ there is a $\gamma' \in Q(T)$ such that for some $x \in \gamma$ and $y \in \gamma'$ we have

$$d(f^i(x), f^i(y)) < \epsilon(T) \text{ for all } 0 \le i < T.$$

Proposition 4. Let $f^i : M \to M$ be an Anosov flow on a compact Riemannian manifold. Let $g : M \to \mathbb{R}$ a Hölder continuous function, let $P(T)$ be the set of all closed orbits of length T, let Q be the collection of all closed orbits for which

$$\int_\gamma g = 0,$$

and let $Q(T) := Q \cap P(T)$. If $Q(T)$ is $\epsilon(T)$-dense in $P(T)$ with $\epsilon(T) \to 0$ as $T \to \infty$, then g vanishes on all closed orbits.

Proof. The idea is to combine the proofs of Proposition 2 and Proposition 3. Take γ a closed orbit of length T. For every $k \geq 1$ and for some $x \in \gamma$, we have

$$\left|\int_\gamma g\right| = \left|\int_0^T g(f^i(x))dt\right| = \frac{1}{k}\left|\int_0^{kT} g(f^i(x))dt\right|.$$

Fix $\epsilon > 0$ arbitrary. Using specification, for $s' = kT + M_\epsilon$ we can find a closed orbit of length s satisfying $|s - s'| < \epsilon$ so that $d(f^i(x), f^i(y)) < \epsilon$ for $0 \leq i \leq kT$. Choosing k sufficiently large, we can find an orbit in $Q(s)$ so that $d(f^i(y), f^i(z)) < \epsilon$, here using the fact that $\epsilon(s) \to 0$ as $k \to \infty$. Combining everything, we have

$$\begin{aligned}\left|\int_\gamma g\right| = \frac{1}{k}\left|\int_0^{kT} g(f^i(x))dt\right| &= \frac{1}{k}\left|\int_0^{kT} g(f^i(x))dt - \int_0^s g(f^i(z))dt\right| \\ &= \frac{1}{k}\left|\int_0^{kT} [g(f^i(x))g(f^i(z))]dt - \int_{kT}^s g(f^i(z))dt\right| \\ &\leq \frac{1}{k}\left[\int_0^{kT} |g(f^i(x)) - g(f^i(z))|dt + \int_{kT}^s |g(f^i(z))|dt\right].\end{aligned}$$

Now use the triangle inequality and the Hölder property to get

$$\begin{aligned}|g(f^i(x)) - g(f^i(z))| &\leq |g(f^i(x)) - g(f^i(y))| + |g(f^i(y)) - g(f^i(z))| \\ &\leq C[d(f^i(x), f^i(y))^\alpha + d(f^i(y), f^i(z))^\alpha] < 2C\epsilon^\alpha.\end{aligned}$$

So we have

$$\left|\int_\gamma g\right| < 2TC\epsilon^\alpha + \frac{1}{k}(M_\epsilon + \epsilon)N,$$

where again N is such that $|g| \leq N$. Let k tend to infinity and ϵ tend to zero to get the desired result. □

Remark.

(1) Comparing this to Proposition 2 it may seem like an improvement, however keep in mind we've modified the definition of ϵ-dense for this scenario. Before it applied to just points, here it applies to orbits.

(2) The same type of argument should apply to Anosov diffeomorphisms.

4. Applications

4.1. **Orbit equivalence for hyperbolic flows.** We discuss the rigidity of flow equivalences for hyperbolic flows following Section 19.2 [7].

We discussed in Section 2.2 how flow equivalence was an extremely strong statement. However, for hyperbolic flows, it turns out that we get flow equivalences for free as long as the periods of periodic orbits line up.

Theorem 7 (Theorem 19.2.9 [7])**.** If $f^t : M \to M$, $g^t : N \to N$ are smooth flows on compact Riemannian manifolds that are orbit equivalent on hyperbolic sets Λ_f, Λ_g respectively, and the periods of the corresponding periodic orbits in Λ_f and Λ_g agree, then f^t and g^t are flow equivalent.

To prove this, we will need the fact that we can improve orbit equivalences between hyperbolic flow to Hölder orbit equivalences between hyperbolic flows.

Theorem 8 (Theorem 6.4.3 [3], Theorem 19.1.5 [7])**.** Let $f^t : M \to M$, $g^t : N \to N$ be flows, $\Lambda_f \subseteq M$ and $\Lambda_g \subseteq N$ compact hyperbolic sets for the flows. Suppose f^t and g^t are orbit equivalent via $h : \Lambda_f \to \Lambda_g$. Then there exists a Hölder continuous h_0 which is arbitrarily C^0 close to h and whose inverse is also Hölder continuous.

Sketch of proof. The idea is to do the construction locally on smooth transversals, then extend it via flow boxes and averaging. After some slight adjustments, the result follows. □

With this, we are now able to prove Theorem 7.

Proof of Theorem 7. Recall we have $f^t : M \to M$, $g^t : N \to N$ smooth flows on compact Riemannian manifolds which are orbit equivalent on hyperbolic sets $\Lambda_f \subseteq M$ and $\Lambda_g \subseteq N$, and furthermore the periods of the corresponding periodic orbits in Λ_f and Λ_g agree. The goal is to show that f^t and g^t are flow equivalent. We break this up into steps.

Step 1: Assume $f^t, g^t : \Lambda \to \Lambda$ Anosov flows, g^t a time change of f^t via Hölder continuous h. Assume as well that all periodic orbits are the same. Since g^t a time change of f^t, we can write

$$g^t(x) = f^{\alpha(t,x)}(x)$$

for a cocycle α. This is trivial if there is a function $\beta : \Lambda \to \mathbb{R}$ differentiable along orbits such that

$$t - \alpha(t, x) = \beta(g^t(x)) - \beta(x).$$

We win if we can show that $t - \alpha(t, x)$ is a coboundary. Suppose x is a periodic point for g^t with period T. Then we have that $\alpha(T, x) = T$, and so $T - \alpha(T, x) = 0$. This holds for every periodic point, so by Theorem 6 we get that this is a coboundary. Thus f^t and g^t are actually flow equivalent.

Step 2: By the Theorem 8, we may assume that $h : \Lambda_f \to \Lambda_g$ is Hölder continuous along with its inverse. Thus $\chi^t = h \circ f^t \circ h^{-1}$ is a Hölder continuous time change of g^t. Using Step 1, we get that χ^t and g^t are flow equivalent, which means that f^t and g^t are flow equivalent.

□

4.2. **Marked length spectrum.** We discuss marked length spectrum for surfaces and how proofs on marked length spectrum use Livschitz, following [18].

Let (S, g) be a surface with negative sectional curvature. We begin with the following well-known fact.

Theorem 9. Let $\mathcal{C}$ denote the set of free homotopy classes. For every $[\gamma] \in \mathcal{C}$ there is a unique closed geodesic.

Problem 13. Prove the above theorem (it is useful to use the classification of isometries on the Poincaré disk).

Using this theorem, we can define the **marked length spectrum** function (abbreviated MLS from here on out),

$$l_g : \mathcal{C} \to [0, \infty), \quad l_g([\gamma]) := \text{ length of closed geodesic.}$$

We can also define the **raw length spectrum** (abbreviated RLS from here on out) as

$$\text{RLS} := \{l_g([\gamma]) : [\gamma] \in \mathcal{C}\}.$$

A natural question to ask is whether these geodesics determine our surface, up to isometry. Milnor gave an example that the RLS does not determine the shape by using a theorem due to Witt, see

[14]. Broadly speaking, this says that just knowing all of the lengths does not determine the shape.[6] The analogous question involving the MLS remained open. In 1978 Guillemin and Khazdan made a big step towards a general spectral rigidy result for negatively curved surfaces by proving that if the MLS are equal and you can deform the metrics smoothly to one another, then they are isometric, see [5].

Theorem 10 (Guillemin and Kazhdan (1978), Theorem 1 [5])**.** Let (S, g) be a compact negatively curved surface. If (S, g') is the surface equipped with a different metric, and (g_t) for $0 \leq t \leq 1$ is a family of metrics depending smoothly on t with $g_0 = g$ and $g_1 = g'$. If $l_g = l_{g_t}$ for every t, then there is a family $\varphi_t : S \to S$ depending smoothly on t such that $g_t = \varphi_t^*(g)$. In other words, (S, g) and (S, g') are isometric.

Croke [1] and Otal [15] both solved the general statement in 1990.[7]

Theorem 11 (Otal (1990) and Croke (1990), Theorem 0.1 [18])**.** If S and S' are closed, negatively curved surfaces with the same marked length spectrum, then S is isometric to S'.

According to Wilkinson, Otal did not originally use the Livschitz theorem, however one can use the Livschitz theorem to simplify things greatly. Following Wilkinson's idea, one breaks up the proof in two steps. The first step is to find $h : T^1\tilde{S} \to T^1\tilde{S}'$ which conjugates geodesic flow. Here is where Theorem 7 is used along with the fact that $l_g = l_{g'}$. The next step is to show that we can use this conjugacy to define an isometry $h : \tilde{S} \to \tilde{S}'$, which then descends to an isometry $h : S \to S'$. This then completes the proof.

The question for general manifolds is still open.

[6] One interesting question to ask is whether it determines it *locally*. I'm not so sure about the answer.

[7] Although both Croke and Otal proved the above theorem, the result is generally known as Otal's theorem.

5. Appendix

5.1. **More cocycle results.** We now give some more results which didn't fit Section 2 but are interesting.

Let $f : X \to X$ be a homeomorphism with X a compact metric space.

Theorem 12 (Theorem 2.9.3 [7])**.** *If $\alpha(n,x)$ is bounded uniformly in n and x, then α is a coboundary with solution given by*

$$\varphi(x) = \sup_{n\in\mathbb{N}} \left\{ -\sum_{i=0}^{n} a(f^i(x)) \right\}.$$

Moreover the solution is measurable.

Remark. It is not necessarily unique.

Remark. The following example shows it need not be continuous. Take the left shift map restricted to the set

$$B_2 = \left\{ \omega = (\omega_j) \in \Sigma_2 : \forall m,n \in \mathbb{Z}, m > n, \left| \sum_{i=n}^{m} (-1)_i^{\omega} \right| \le 2 \right\}.$$

Consider $g : B_2 \to \mathbb{R}$ given by $g(\omega) = (-1)^{\omega_0}$. This determines a cocycle which is a coboundary, and has solution

$$\varphi(x) = \sup_{n\in\mathbb{N}} \left\{ -\sum_{i=0}^{n} (-1)^{\omega_i} \right\}.$$

This is bounded and Borel measurable, but not continuous by the fact that $\sigma_2|_{B_2}$ is not conjugate to any topological Markov chain. The discontinuities then arise from points where the preimage has more than one point in a semiconjugacy.

Theorem 13 (Gottschalk, Hedlund (1955), Theorem 2.9.4 [7])**.** *If $f : X \to X$ a minimal*[8] *continuous map, $g : X \to \mathbb{R}$ a continuous map such that for all $M > 0$ there is an n and $x_0 \in X$ so that*

$$\left| \sum_{i=0}^{n} g(f^i(x_0)) \right| \le M$$

then the cocycle generated by g is a coboundary.

5.2. **Solutions.** Here we give solutions to some of the problems.

Solution (Problem 1)**.**

(1) Notice
$$\alpha(0,x) = \alpha(0+0,x) = \alpha(0, f^0(x)) + \alpha(0,x) = 2\alpha(0,x).$$
We deduce $\alpha(0,x) = 0$.

(2) We see
$$0 = \alpha(0,x) = \alpha(n+(-n),x) = \alpha(-n,x) + \alpha(n, f^{-n}(x)).$$
Subtracting gives the result.

Solution (Problem 2)**.** Remember we just need to show the cocycle condition. We have

$$\begin{aligned} \alpha(n+m,x) = \log((f^{n+m})'(x) &= \log(f'(f^{n+m-1}(x))f'(f^{n+m-2}(x))\cdots f'(x)) \\ &= \log(f'(x)\cdots f'(f^{n-1}(x))) + \log(f'(f^n(x))\cdots f'(f^{n+m-1}(x))) \\ &= \log((f^n)'(x)) + \log((f^m)'(f^n(x))) = \alpha(n,x) + \alpha(m, f^n(x)). \end{aligned}$$

[8] The orbit of every point is dense.

Solution (Problem 3)**.** Let α, β be two cocycles over f.

(1) Define
$$\gamma(t,x) = \alpha(t,x) + \beta(t,x).$$
Then
$$\gamma(t+s,x) = \alpha(t+s,x) + \beta(t+s,x)$$
$$= [\alpha(t,x) + \beta(t,x)] + [\alpha(s,f^t(x)) + \beta(s,f^t(x))] = \gamma(t,x) + \gamma(s,f^t(x)).$$

(2) Let $a \in \mathbb{R}$, define
$$\gamma(t,x) = a\alpha(t,x).$$
Then
$$\gamma(t+s,x) = a\alpha(t+s,x) = a[\alpha(t,x) + \alpha(s,f^t(x))] = \gamma(t,x) + \gamma(s,f^t(x)).$$

(3) Let $h : X \to X$ be a homeomorphism. Define
$$\gamma(t,x) = \alpha(t,h(x)).$$
Then
$$\gamma(t+s,x) = \alpha(t+s,h(x)) = \alpha(t,h(x)) + \alpha(s,f^t(h(x)))$$
$$= \alpha(t,h(x)) + \alpha(s,h(f^t(x))) = \gamma(t,x) + \gamma(s,f^t(x)).$$

Solution (Problem 4)**.** Our proof that $\alpha(0,x) = 0$ relied on the fact that we do not live in some characteristic two space. Similarly our proof on $\alpha(-n,x)$ relied on $\alpha(0,x)$. The bijection still holds, however we need to be careful about the order of things if we're not commutative.

Solution (Problem 5)**.** This follows by the chain rule.

Solution (Problem 6)**.** Since α is a coboundary, we know
$$\alpha(t,x) = \varphi(f^t(x)) - \varphi(x).$$
Thus if x is periodic with period T, then
$$\alpha(T,x) = \varphi(f^T(x)) - \varphi(x) = \varphi(x) - \varphi(x) = 0,$$
and simultaneously
$$\alpha(T,x) = \int_0^T a(f^t(x))dt.$$

Solution (Problem 7)**.** The fact that $\alpha(t,x) \geq 0$ follows since f and g share the same orientation. If x is not periodic, $g^t(x) \neq x$ for $t > 0$, so $\alpha(t,x) > 0$.

Solution (Problem 8)**.** We have α is a cocycle over g, t is clearly a cocycle over g, so the result follows.

Solution (Problem 9)**.** On a compact Riemannian manifold, all metrics are equivalent in the sense that if $\|\cdot\|$ and $\|\cdot\|'$ are two norms generated by a metric, then there exist $C_1, C_2 > 0$ so that
$$C_1\|\cdot\| \leq \|\cdot\|' \leq C_2\|\cdot\|.$$
Once we have this fact the result follows. To see this fact, take $\gamma : T^1M \to \mathbb{R}$ defined by
$$\gamma(v) = \frac{\|v\|'}{\|v\|},$$
where we assume the norm $\|\cdot\|$ is on TM and
$$T^1M = \{v \in TM : \|v\| = 1\}$$
is with respect to this. This is continuous and since things are compact we have constants $C_1, C_2 > 0$ so that $C_1 \leq \gamma \leq C_2$.

Solution (Problem 10)**.** All of the information can be found in Section 1.4 [7].

Solution (Problem 11)**.** Suppose $g = \varphi \circ f - \varphi$ and $x \in M$ satisfies $f^n(x) = x$. Then

$$\begin{aligned}
\sum_{i=0}^{n-1} g(f^i(x)) &= \sum_{i=0}^{n-1} [\varphi(f^{i+1}(x)) - \varphi(f^i(x))] \\
&= \sum_{i=1}^{n} \varphi(f^i(x)) - \sum_{i=0}^{n-1} \varphi(f^i(x)) \\
&= \sum_{i=1}^{n-1} \varphi(f^i(x)) + \varphi(f^n(x)) - \sum_{i=0}^{n-1} \varphi(f^i(x)) \\
&= \sum_{i=1}^{n-1} \varphi(f^i(x)) + \varphi(x) - \sum_{i=0}^{n-1} \varphi(f^i(x)) \\
&= \sum_{i=0}^{n-1} \varphi(f^i(x)) - \sum_{i=0}^{n-1} \varphi(f^i(x)) = 0.
\end{aligned}$$

Solution (Problem 12)**.** All of the information can be found in Section 6.3 [3].

Solution (Problem 13)**.** The proof of this theorem can be found in [2].

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