RYLL-NARDZEWSKI'S THEOREM

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Contents

1. The Ryll-Nardzewski Theorem

This section heavily follows Granas and Dugundji [2].

1.1. **What is Ryll-Nardzewski.**

The Ryll-Nardzewski theorem highlights the interplay between the natural topology of a locally convex space and its weak topology. Essentially one is able to extract information on whether there is a fixed point using only weak topology information. Some major applications of the Ryll-Nardzewski theorem are the construction of a Haar measure on a compact group, the existence of a left-invariant mean on $W(G)$ (the space of weakly periodic functions), and the existence of invariant linear functionals under the action of a group of isometries. We present some of these here.

1.2. **Preliminaries.**

We need a few definitions before diving in.

Definition (Fixed Point)**.** If $\mathcal{F}$ is a family of maps of a space X into itself, a *fixed point for* $\mathcal{F}$ is a point $x_0 \in X$ so that for all $f \in \mathcal{F}$, we have $f(x_0) = x_0$.

Definition (Noncontracting)**.** Let $\mathcal{F}$ be a family of self-maps of a set X in some linear topological space. The family $\mathcal{F}$ is called *noncontracting* on X if for any distinct points $x, y \in X$, zero does not belong to the closure of the set

$$\{Tx - Ty : T \in \mathcal{F}\}.$$

Definition (Distal)**.** The family $\mathcal{F}$ of self-maps of a set X in some linear topological space is called *distal* on X if for any distinct $x, y \in X$ there is an open covering $\{V_\alpha\}$ of X such that

$$Ty \notin \bigcup_\alpha \{V_\alpha : Tx \in V_\alpha\} \text{ for each } T \in \mathcal{F}.$$

The last two definitions will essentially be equivalent in the setting of a locally convex space, as seen in the next lemma.

Lemma 1 (**Lemma 9.2** [2])**.** Suppose E is a locally convex space, $X \subset E$ is compact, and let $\mathcal{F}$ be a family of self-maps of X. The following are equivalent:

(1) $\mathcal{F}$ is distal on X.
(2) For each net $\{T_\beta\} \subset \mathcal{F}$ and any pair of distinct points $x, y \in X$, if $T_\beta x \to u$ and $T_\beta y \to v$, then $u \neq v$.
(3) $\mathcal{F}$ is noncontracting on X.

Proof. (1) $\implies$ (2): Suppose $\mathcal{F}$ is distal on X, let $\{T_\beta\} \subset \mathcal{F}$ be a net, and suppose $T_\beta x \to u$ and $T_\beta y \to v$ for distinct $x, y \in X$. Suppose for contradiction $u = v$. Consider a cover $\{V_\alpha\}$ of X, and consider γ so that $u \in V_\gamma$. Since $T_\beta x \to u$ and $T_\beta y \to u$, we get that V_γ contains almost every $T_\beta x$ and $T_\beta y$. In particular, this means for some β we have

$$T_\beta x \in V_\gamma \subset \bigcup_\alpha \{V_\alpha : T_\beta y \in V_\alpha\}.$$

This contradicts the fact that $\mathcal{F}$ is distal.
(2) $\implies$ (3): The goal is to show that $\mathcal{F}$ is noncontracting. Assume for contradiction

$$0 \in \overline{\{Tx - Ty : T \in \mathcal{F}\}}.$$

This means that we can construct a net $\{T_\beta\} \subset \mathcal{F}$ so that $T_\beta x - T_\beta y \to 0$. Since X is compact, we can refine this so that $T_\beta x \to u$ and $T_\beta y \to v$. This implies that $u = v$, contradicting (2).
(3) $\implies$ (2): If $x, y \in X$ are distinct, then being noncontracting says that

$$0 \notin \overline{\{Tx - Ty : T \in \mathcal{F}\}}.$$

In particular, there is a neighborhood of the origin containing no $Tx - Ty$ for all $T \in \mathcal{F}$. Choose a balanced neighborhood $U \subset W$ with $U - U \subset W$. Shifting is a homeomorphism, so

$$\{X \cap (U + p) : p \in X\}$$

is an open cover of X. In particular, there is no $T \in \mathcal{F}$ so Tx and Ty belong to a common $U + p$, for if there were then

$$\{Tx, Ty\} \subset U + p \implies Tx - Ty = (Tx - p) - (Ty - p) \in U - U \subset W,$$

which is impossible. Thus we have that $\mathcal{F}$ is distal on X. □

Definition ($\mathcal{F}$-invariant)**.** If $\mathcal{F}$ is a family of self-maps of X, a subset $A \subset X$ is called *$\mathcal{F}$-invariant* if $T(A) \subset A$ for all $T \in \mathcal{F}$.

Definition (Minimal closed $\mathcal{F}$-invariant subset)**.** A closed nonempty $A \subset X$ that is $\mathcal{F}$-invariant and has no proper closed $\mathcal{F}$-invariant subset is called a *minimal closed $\mathcal{F}$-invariant subset.*

Denote by $\operatorname{co}(A)$ the convex hull of a set A, denote by $\overline{\operatorname{co}}(A)$ the convex closure, and denote by $E(A)$ the set of extreme points of A. We will utilize an extended version of Krein-Milman in locally convex spaces. That is, we will use the following.

Theorem 1 (**Theorem 3.24, 3.25** [4])**.** Let E be a locally convex space, $A \subset E$.
(1) If $\overline{\operatorname{co}}(A)$ is compact, then $\overline{\operatorname{co}}(A)$ has extreme points.
(2) If A is also compact, then $E(\overline{\operatorname{co}}(A)) \subset A$.

Utilizing this result, we get the following.

Theorem 2 (**Theorem 9.3** [2])**.** Let C be a nonempty compact convex set in a locally convex space E, and let $\mathcal{F}$ be a semigroup of continuous affine maps of C into itself. If $\mathcal{F}$ is distal on each minimal closed $\mathcal{F}$-invariant set, then $\mathcal{F}$ has a fixed point.

Proof. We break this up into four steps.

Step 1: We first claim there is a minimal nonempty compact convex subset that is $\mathcal{F}$-invariant. To do this, we use Zorn's Lemma.

Proof of Step 1. Let $\mathcal{G}$ be the collection of all nonempty compact convex subsets that are $\mathcal{F}$-invariant. Note that $\mathcal{G}$ is nonempty, since $C \in \mathcal{G}$. This set is partially ordered by inclusion, and if $\{K_\alpha\} \subset \mathcal{G}$ is a desecending chain then $\bigcap K_\alpha \in \mathcal{G}$ is a lower bound. By Zorn's Lemma, we get a minimal $C_0 \subset C$ in $\mathcal{G}$. □

Step 2: Next, we claim there is a smallest nonempty compact subset of C_0 that is $\mathcal{F}$-invariant. To prove this, we again use Zorn's Lemma.

Proof of Step 2. Let $\mathcal{G}_0$ be the collection of all nonempty compact subsets of C_0. Note $C_0 \in \mathcal{G}_0$, so it is nonempty. Again, this has a partial ordering given by inclusion, and again if we have a descending chain $\{K_\alpha\} \subset \mathcal{G}_0$, then $\bigcap K_\alpha \in \mathcal{G}_0$ is a lower bound. So we have a minimal $X \subset C_0$. □

Step 3: We now claim that X has one point.

Proof of Step 3. We proceed by contradiction. Assume $x, y \in X$ are such that $x \neq y$. Since C_0 is convex, we get $(x+y)/2 \in C_0$. Since C_0 is $\mathcal{F}$-invariant,

$$A = \left\{ T\left(\frac{x+y}{2}\right) : T \in \mathcal{F} \right\} \subset C_0.$$

Note three things about A:

(a) If we take the closure of A, we have $\overline{A} \subset C_0$.

(b) We have that $\overline{A}$ is $\mathcal{F}$-invariant.

(c) Since each T is affine, we have

$$\text{co}(\overline{A}) \subset C_0$$

is also compact.

Since C_0 is minimal, we get that $\text{co}(\overline{A}) = C_0$.

Let $z \in E(C_0)$ be an extreme point. Since $\overline{A}$ is compact, the extended Krein-Milman theorem says that $z \in \overline{A}$. So we can find a net $T_\alpha((x+y)/2) \to z$. We have $T_\alpha x$ and $T_\alpha y$ are both in the compact set X, so assume $T_\alpha x \to u$ and $T_\alpha y \to v$, both in X. Then

$$z = \lim \frac{T_\alpha x + T_\alpha y}{2} = \frac{u+v}{2}.$$

Since z is an extreme point, $u = v = z$. Let $\{V_\alpha\}$ be an open cover of X, and let β be such that $u \in V_\beta$. Then almost every $T_\alpha x$, $T_\alpha y \in V_\beta$, contradicting the fact that $\mathcal{F}$ is distal on X. This tells us that X must only have one point. □

Step 4: Since $X = \{x_0\}$ has one point and X is $\mathcal{F}$-invariant, we see that $T(x_0) = x_0$ for all $T \in \mathcal{F}$. This forces x_0 to be a fixed point for $\mathcal{F}$.

□

Corollary 1 (**Theorem 9.4** [2])**.** Let C be a compact convex subset of a locally convex space E, and let $\mathcal{F}$ be a semigroup of continuous affine self-maps of C. If $\mathcal{F}$ is distal on C, then $\mathcal{F}$ has a fixed point.

Proof. Let $X \subset C$ be a closed subset. Then we have that X is compact. We claim that $\mathcal{F}$ being distal on C implies $\mathcal{F}$ is distal on X. We use **Lemma 1** to see this. Let $x, y \in X \subset C$ be distinct points, $\{T_\beta\} \subset \mathcal{F}$ a net, and suppose $T_\beta x \to u$ and $T_\beta y \to v$. Since $x, y \in C$, we have that $u \neq v$. Since this applies for each net and every pair of distinct points in X, we get that $\mathcal{F}$ is distal on X.

Since $\mathcal{F}$ is a self-map of C, any minimal closed $\mathcal{F}$-invariant set will be contained in C, and so $\mathcal{F}$ must be distal on each minimal closed $\mathcal{F}$-invariant set by the above observation. Using **Theorem 2**, we get that $\mathcal{F}$ has a fixed point. □

1.3. **The Theorem.** We can now present the Ryll-Nardzewski theorem as a generalization of **Theorem 1**.

Theorem 3 (**Theorem 9.6** [2])**.** Let C be a nonempty weakly compact convex set in a locally convex space E. Let $\mathcal{F}$ be a semigroup of weakly continuous affine self-maps of C. If $\mathcal{F}$ is strongly noncontracting on C, then $\mathcal{F}$ has a fixed point.

Proof. Let $\text{Fix}(T) = \{x \in E : T(x) = x\}$ be the collection of fixed points for a map T. Then the collection of fixed points for the family $\mathcal{F}$ can be expressed as

$$A := \bigcap \{\text{Fix}(T) : T \in \mathcal{F}\}.$$

The goal is to show that $A \neq \varnothing$. Like before, we break this into a few steps.

Step 1: We note that $\text{Fix}(T)$ is weakly closed, hence weakly compact. By the finite intersection property, it suffices to show that finite intersections of $\text{Fix}(T)$ are nonempty. Doing so, we can deduce A is nonempty. Let $T_1, \dots, T_n \in \mathcal{F}$ and let $\mathcal{G} = \langle T_1, \dots, T_n \rangle$ the semigroup generated by the T_j. Note that $\mathcal{G}$ is countable. If we show $\mathcal{G}$ has a fixed point, then $\bigcap_{j=1}^n \text{Fix}(T_j) \neq \varnothing$ and we are done.

Step 2: Pick $c_0 \in C$ and consider

$$Q = \overline{\text{co}}\{T(c_0) : T \in \mathcal{G}\}.$$

Because $\mathcal{G}$ is countable, Q is strongly separable. Because each T is affine, Q is $\mathcal{G}$-invariant, and since Q is a closed convex subset of C, it is weakly closed and hence weakly compact. So it is enough to prove it for Q and $\mathcal{G}$. Relabeling, we may assume C is Q and $\mathcal{G}$ is $\mathcal{F}$. We get the additional assumption that C is strongly separable.

Step 3: The goal is to show $\mathcal{F}$ is weakly distal on each weakly closed minimal $\mathcal{F}$-invariant set $X \subset C$. Let X be such a set, and suppose $x \neq y$ are distinct in X. By assumption, $\mathcal{F}$ is strongly noncontracting, so there exists a strongly open neighborhood of the origin V so that

$$V \cap \{Tx - Ty : T \in \mathcal{F}\} = \varnothing.$$

Choose W convex so that $\overline{W} - \overline{W} \subset V$. Then $\overline{W}$ is a strongly closed convex body, and since C is strongly separable, a countable number of translates $\overline{W}_i = \overline{W} + x_i$ cover X. Each $\overline{W}_i$ is strongly closed and convex, so they are also weakly closed. Hence $\{X \cap \overline{W}_i\}$ is a countable weakly closed cover of the weakly compact set X. By Baire's theorem, at least one of these sets contains a weakly open set (must have nonempty interior). Let $U \subset X \cap (\overline{W} + x_0)$ be the weakly open nonempty set.

Step 4: If we show that the family $\{T^{-1}(U) : T \in \mathcal{F}\}$ satisfies the distal property for $\mathcal{F}$, we win by applying **Theorem 2** to find our fixed point. Notice that these sets must cover X, since otherwise

$$X \setminus \bigcup \{T^{-1}(U) : T \in \mathcal{F}\}$$

would be a weakly compact $\mathcal{F}$-invariant proper subset of X, contradicting the minimality of X. Next, we note that for no $S \in F$ do we have Sx and Sy belonging to a common $T^{-1}(U)$. Otherwise we have TSx and TSy would belong to $UX \cap (\overline{W} + x_0)$ so that $TSx - TSy \in \overline{W} - \overline{W} \subset V$, and since $TS \in \mathcal{F}$ and $\mathcal{F}$ strongly noncontracting this would contradict our choice of V. Thus, it is indeed distal, showing $\mathcal{F}$ is weakly distal on X.

□

2. Applications

2.1. **Weakly almost periodic functions.** This section will heavily follow Burckel [1].

Let G be a locally compact abelian topological group G. Denote by $C(G)$ the space of bounded complex-valued continuous functions $x(t)$ on G under the norm

$$\|x\| = \text{supp}_{t\in G}|x(t)|.$$

Definition. We call a function $f \in C(G)$ *weakly almost periodic* (denoted $f \in W(G)$) if its orbit

$$\mathcal{O}(f) = \{L_x f : x \in G\}$$

is relatively compact with respect to the weak topology in $C(G)$, where

$$L_x(f)(y) = f(xy).$$

The goal here is to show that $W(G)$ admits a left-invariant mean. We give a few more definitions before jumping into the main result.

Definition (Stationary)**.** Let $CO(f) := \overline{\text{co}(\mathcal{O}(f))} = \overline{\text{co}}(\overline{\mathcal{O}(F)})$ be the (weak) closure of the convex hull of the orbit of f. G is said to be $W(G)$-*stationary* if for each $f \in W(G)$, $\overline{\text{co}}(\mathcal{O}(F))$ contains a constant function.

Definition (Invariant Mean)**.** For G a locally compact abelian topological group, A a norm closed subspace of $C(G)$, an *invariant mean* on A is any linear functional M on A satisfying

(1) $M \neq 0$ and $M(1) = 1$ if $1 \in A$.
(2) $f \in A$, $f \geq 0$ implies $M(f) \geq 0$.
(3) $M(L_x f) = M(f)$ for all $x \in G$, $f \in A$.

Definition (Amenable)**.** For G a locally compact abelian topological group, A a norm closed subspace of $C(G)$, we say that A is *amenable* if there is an invariant mean.

We assume the results of the following theorems.

Theorem 4 (**Theorem A.21** [1])**.** If X is a Banach space, $K \subset X$ is weak compact, then $\overline{\text{co}}(K)$ is also weak compact.

Theorem 5 (**Theorem 1.25** [1])**.** For G a locally compact abelian topological group, the following two statements are equivalent.

(1) G is $W(G)$-stationary.
(2) $W(G)$ is amenable.

Assuming this, we can use Ryll-Nardzewski (**Theorem 3**) to say the following.

Theorem 6 (**Corollary 1.26** [1])**.** If G is a locally compact abelian topological group, then $W(G)$ has an invariant mean. In other words, $W(G)$ is amenable.

Proof. Let $f \in W(G)$. We claim that $CO(f)$ is weakly compact. Note that by definition $\overline{\mathcal{O}(f)}$ is compact, so $\overline{\text{co}}(\overline{\mathcal{O}(f)}) = CO(F)$ is compact by **Theorem 4**. Since G is a group, each R_x is a linear isometry on $C(G)$. Hence $\{R_x : x \in G\}$ acts noncontractively and weakly continuously on the weak compact convex set $CO(f)$. Applying **Theorem 3**, there exists an $h \in CO(f)$ which is invariant under all R_x. So $h(e) = R_x h(e) = h(x)$ for all $x \in G$. So h is constant, and therefore $CO(f) \cap \mathbb{C} \neq \varnothing$. This tells us that G is $W(G)$-stationary, and applying the **Theorem 5** tells us that $W(G)$ has an invariant mean M. ☐

Remark. This shows that for every locally compact abelian topological group, the space of weakly almost periodic functions is amenable.

2.2. **Construction of a Haar measure.** This section will heavily follow Kiesenhofer [3].

G now denotes a compact topological Hausdorff group. We write $\cdot'$ to denote the topological dual versus the algebraic dual $\cdot^*$.

Definition (Haar measure)**.** A *Haar measure* on G is a measure μ on the Borel sets of G which satisfies the following:

(1) We have that μ is a Radon measure (inner regular and finite on compact sets).
(2) We have that μ is invariant under translation:
$$\mu(Ag) = \mu(A) = \mu(gA)$$
for all Borel sets $A \subset G$ and elements $g \in G$.

The goal is to establish the existence of a Haar measure for such a G. That is, to prove the following theorem.

Theorem 7. If G is a compact topological Hausdorff group, then G admits a Haar measure.

Note that since μ is a Radon measure, meaning finite on compact sets, we can normalize μ so that $\mu(G) = 1$. So without loss of generality we can assume μ is a probability measure. Examine the space
$$Q := \{\mu : \mu \text{ is a Radon measure and } \mu(G) = 1\}.$$
If our Haar measure μ exists, we have that $\mu \in Q$. Furthermore, μ must be fixed under the mappings
$$\mathcal{F} = \{R_g : g \in G\} \cup \{L_g : g \in G\}$$
where
$$R_g(\mu)(A) = \mu(Ag),$$
$$L_g(\mu)(A) = \mu(gA).$$
Denote by $C(G)$ the set of continuous complex valued functions on G. We recall the Riesz-Representation theorem.

Theorem 8 (**Theorem 3.1** [3])**.** Let G be a locally compact Hausdorff space. The mapping
$$\mu \mapsto I_\mu := \int_G \cdot d\mu$$
is a bijection from the set of Radon measures on G to the set of positive linear functionals on $C_c(G)$.

View $\widehat{Q} := \Phi(Q) \subset C_c(G)^*$. Since G is compact, all functions have compact support, so $C_c(G) = C(G)$. Since every Radon measure μ on G is finite, we have that the corresponding functional I_μ is continuous on $C(G)$ with respect to the supremum norm. So $\widehat{Q} \subset C(G)'$ (the topological dual). So we can write
$$\widehat{Q} = \{I \in C(G)' : I \text{ is positive and } I(1) = 1\}.$$
Let $\mathrm{Eval}(f) : C(G)' \to \mathbb{C}$ denote
$$\mathrm{Eval}(f)(I) = I(f).$$
This is a linear funcitonal, and we see that we can express $\widehat{Q}$ as
$$\widehat{Q} = \overline{B(C(G)')} \cap \mathrm{Eval}_1^{-1}(1) \cap \bigcap_{f \geq 0} \mathrm{Eval}_f^{-1}(\mathbb{R}_0^+),$$
where $B := B(C(G)')$ is the unit ball in $C(G)'$. By definition of the weak* topology and Banach-Alaoglu, we get that $\widehat{Q}$ is weakly compact and convex.

We now need to translate the problem in terms of the topological dual of $C(G)$ now. We had before that a measure $\mu \in Q$ is a fixed point for the family $\mathcal{F}$ iff it is a Haar measure. Now if μ is a Haar measure, we have that I_μ needs to be a fixed point of the map

$$\widehat{F} = \{\widehat{R}_x : x \in G\} \cup \{\widehat{L}_x : x \in G\},$$

where

$$\widehat{R}_x(I_\mu)(f) = \int_G f(xz) d\mu(z)$$

for $f \in C(G)$. Let $S = \langle \widehat{F} \rangle$ be the semigroup generated by $\widehat{F}$.

Finally, we observe a nice lemma.

Lemma 2 (**Lemma 3.2** [3])**.** Let G be a compact group, $I \in \widehat{Q} \subset C(G)'$. Then

$$\rho : G \times G \to C(G)' : (g, h) \mapsto \widehat{R}_g \widehat{L}_h(I)$$

is continuous.

We now have all of the tools to prove our theorem.

*Proof of **Theorem 7**.* To use **Theorem 3**, we need to show that the elements in $\mathcal{S}$ are weakly continuous affine self-maps of Q which are (strongly) noncontracting on C. First, let's show that the elements in $\mathcal{S}$ are affine. Take $\widehat{R}_x \in \mathcal{S}$ (the argument will be analogous for $\widehat{L}_x$). Consider $(t_i)_{i=1}^n \subset [0,1]$ with $\sum t_i = 1$, $I_i \in \widehat{Q}$. Then

$$\widehat{R}_x \left(\sum_{i=1}^n t_i I_i \right) (f) = t_i \sum_{i=1}^n \int f(xz) d\mu_i(z) = t_i \sum_{i=1}^n \widehat{R}_x(I_i).$$

Thus the elements are affine, since compositions will preserve this property.

Next, observe that every $S \in \mathcal{S}$ is continuous. Again, it suffices to check this on $\widehat{R}_x$ (the argument for $\widehat{L}_x$ will be the same). It suffices to check that it is continuous at 0, and to do that we just check via nets. Let $(I_i) \subset \widehat{Q}$ be a net. Then $I_i \to 0$ implies

$$\mathrm{Eval}_f(I_i) = I_i(f) \to 0 \text{ for all } f \in C(G)$$

implies

$$I_i(f(x\cdot)) = R_x I_i(f) \to 0 \text{ for all } f \in C(G)$$

implies $R_x I_k \to 0$. So the map is continuous.

We now show noncontracting. Let $M := \{S(I) - S(J) : S \in \mathcal{S}\}$, $I \neq J$ arbitrary elements in $\widehat{Q}$. The elements of $\mathcal{S}$ are injective, so $0 \notin M$. If we can show M is closed, we are done. We can write

$$M = \{\widehat{R}_x \widehat{L}_y(I) - \widehat{R}_x \widehat{L}_y(J) : x, y \in G\}$$

using the definition of $\mathcal{S}$. Thus we see $\rho(G \times G) = M$, where ρ as in **Lemma 2**. This implies M is closed, and we get that the family is noncontracting on $\widehat{Q}$. We now apply **Theorem 3** to get that there is a Haar measure. □

Remark.

- One can also see that the Haar measure is unique with a nice trick involving Fubini-Tonelli (see [3]).
- As noted in [3], this heavily depends on compactness. We can weaken to locally compact abelian groups using the Markov-Kakutani theorem.

References

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