NOTES ON PDE APPLICATIONS

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Contents

Remark. We will closely follow the notes by Hunter [3], which follows the lecture notes. Thanks to Thomas O'Hare for pointing out typos.

1. The Heat Equation

1.1. **What is the heat equation.** Consider our space as $\mathbb{R}^n$, and let $u(x,t) : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ be a function which measures the *heat*[1] of our space at the point $x \in \mathbb{R}^n$ and at time $t \in [0,\infty)$. The heat equation, then, is a PDE which describes the *flow* of heat in some infinite n-dimensional plate, all made of the same material. The PDE is given by

$$\begin{cases} \frac{\partial u}{\partial t} = \alpha \left(\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} \right) \\ u(x,0) = f(x), \end{cases}$$

where here $f(x)$ denotes our initial condition, and α is some real constant which measures the diffusitivity of heat in the material of our space. To simplify things, we'll just ignore the constant α (that is, set $\alpha = 1$). We also will use the notation $u_t = \partial_t u$ to denote the tth derivative of u, and we will use ∇^2 to denote the *Laplacian operator*; that is,

$$\nabla^2 u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}.$$

With these notational simplifications, our heat equation turns into

$$\begin{cases} \partial_t u = \nabla^2 u \\ u(x,0) = f(x). \end{cases}$$

The derivation of this can be found in many resources (see [1]), but for the purpose of these notes we'll just assume this.

[1] A good question is what is heat, which we will not explore in this.

The goal, then, is to somehow use Fourier theory to find some kind of solution to this PDE. Before beginning, we'll build some language.

Definition. A *differential operator* is a linear partial differential operator; that is, an operator L of the form

$$Lf(x) = \sum_{|\alpha|\leq m} a_\alpha(x)\partial^\alpha f(x), \quad a_\alpha \in C^\infty.$$

Remark. As an aside, we also had the following theorem on differential operators in the lecture notes (left as an exercise).

Theorem. A differential operator L satisfies

$$L(f\circ T) = (Lf)\circ T$$

where T is either a translation or a rotation iff there is a polynomial P in one variable such that $L = P(D)$.

Proof. We first note that L is translation-invariant iff L has constant coefficients. To see this, first assume that L is translation-invariant. Then for all y, we see that

$$\tau_y L = \sum_{|\alpha|\leq m} a_\alpha(x-y) = \sum_{|\alpha|\leq m} a_\alpha(x) = L \text{ for all y } \in \mathbb{R}^n.$$

Hence, we have that $a_\alpha(x-y) = a_\alpha(x)$ for all y, and so therefore we have that $a_\alpha(x) = a_\alpha(0)$ for all x. That is, $a_\alpha(x)$ is a constant function for all α. The backwards direction is clear, since L having constant coefficients implies that $a_\alpha(x-y) = a_\alpha(x)$ for all $y\in\mathbb{R}^n$, and so L is translation invariant.

We introduce now the operator

$$D^\alpha = (2\pi i)^{-|\alpha|}\partial^\alpha.$$

Since the a_α are all constnat here, we introduce the notation $b_\alpha = (2\pi i)^{|\alpha|}a_\alpha$, and we see that we can rewrite L as

$$L = \sum_{|\alpha|\leq m} b_\alpha D^\alpha.$$

Thus, for constant coefficients, we see that we have $L = Q(D)$, where Q is some polynomial in n variables.

Another advantage of introducing this new notation is that for well-behaved f, we have

$$\begin{aligned}(Lf)^\wedge &= \sum_{|\alpha|\leq m} b_\alpha [D^\alpha f]^\wedge \\ &= \sum_{|\alpha|\leq m} b_\alpha \xi^\alpha \widehat{f}.\end{aligned}$$

Hence, $(Lf)^\wedge = Q\widehat{f}$. Now, notice by **Proposition 8.22 (b)** [4] that Fourier transforms commute with rotations. Using these two concepts, we see that L commutes with rotations iff Q is rotation-invariant.

We decompose $Q = \sum_0^m Q_j$ where the Q_j are homogenous of degree j. It follows that Q is rotation-invariant iff each Q_j is rotation-invariant [apparently an easy induction argument, will possibly do in the future]. This means that $Q_j(\xi)$ depends only on $|\xi|$, so that we have $Q_j(\xi) = c_j|\xi|^j$ by homogeneity. Note that $|\xi|^j$ is a polynomial iff j is even, so we see that $c_j = 0$ for j odd. Setting $b_k = (-4\pi^2)^{-k}c_{2k}$, we get that

$$Q(\xi) = \sum b_k(-4\pi^2|\xi|^2)^k,$$

which implies that

$$L = \sum b_k (\nabla^2)^k.$$

□

Notice that for f well-behaved (one example being $f \in \mathcal{S}$, Schwartz space) and for a_α constant, we get that applying the Fourier transform to a differential operator gives us

$$\begin{aligned}(Lf)^\wedge(k) &= \sum_{|\alpha|\le m} [a_\alpha(x)\partial^\alpha f(x)]^\wedge \\ &= \sum_{|\alpha|\le m} a_\alpha (2\pi i k)^\alpha \widehat{f}(m).\end{aligned}$$

Writing out the Laplacian in this form, we have that

$$\nabla^2 = \sum_1^n \partial_j^2,$$

and so we see that for $f \in \mathcal{S}$,

$$\begin{aligned}(\nabla^2 f)^\wedge(k) &= \sum_1^n [\partial_j^2 f]^\wedge(k) \\ = \sum_1^n [\partial_j(\partial_j f)]^\wedge(k) = 2\pi i \sum_1^n k_j [\partial_j f]^\wedge(k) &= -4\pi^2 \sum_1^n k_j^2 \widehat{f}(k) \\ &= -4\pi^2 \widehat{f}(k) \sum_1^n k_j^2 = -4\pi^2 |k|^2 \widehat{f}(k)\end{aligned}$$

where we used **Theorem 8.22 (e)** [4] twice.

Here, we define the *principle symbol* to be the polynomial $P(k) = -4\pi^2|k|^2$. Thus, we write

$$(\nabla^2 f)^\wedge(k) = P(k)\widehat{f}(k).$$

1.2. **Solving the heat equation if everything works.** Turning our attention back to the original PDE, we rewrite the differential equation as

$$\partial_t u - \nabla^2 u = 0,$$

and taking the Fourier transform of both sides (here, noting that we are doing the *formal* Fourier transform, meaning that we don't care about convergence or existence), we have

$$\begin{aligned}[\partial_t u - \nabla^2 u]^\wedge(m) &= [\partial_t u]^\wedge(m) - [\nabla^2 u]^\wedge(m) \\ &= \partial_t \widehat{u}(m) + 4\pi^2 |m|^2 \widehat{u}(m) = 0,\end{aligned}$$

with initial condition given by

$$\widehat{u}(m, 0) = \widehat{f}(m).$$

That is, by applying the Fourier transform, we've changed the problem from a PDE to an ODE. We just need to solve the ODE

$$\begin{cases}\partial_t \widehat{u}(m,t) = -4\pi^2|m|^2 \widehat{u}(m,t) \\ \widehat{u}(m,0) = \widehat{f}(m),\end{cases}$$

assuming that the conditions are all correct in order for this to exist. Such an ODE is easy to solve; we "guess" that our potential function looks something of the form $e^{4\pi^2|m|^2 t}$, and we see that

$$\partial_t \left[\widehat{u}(m,t) e^{4\pi^2|m|^2 t}\right] = 4\pi^2 |m|^2 t \partial_t \widehat{u}(m,t) e^{4\pi^2|m|^2 t} - 4\pi^2 |m|^2 \widehat{u}(m,t) e^{4\pi^2|m|^2 t} = 0,$$

so that

$$\widehat{u}(m,t)e^{4\pi^2|m|^2t} = C(m)$$

for some function $C(m)$. Plugging in $t = 0$, we have

$$\widehat{u}(m,0) = C(m) = \widehat{f}(m)$$

by the initial conditions, and so

$$\widehat{u}(m,t)e^{4\pi^2|m|^2t} = \widehat{f}(m) \implies \widehat{u}(m,t) = \widehat{f}(m)e^{-4\pi^2|m|^2t}.$$

Furthermore, the uniqueness and existence theorem from ODEs tells us that this is a unique solution.

Thus, we've established a unique solution for $\widehat{u}(m,t)$. Since we're in the condition where things are just going to "work," let's take the inverse Fourier transform and see what happens. Doing so, we have

$$u(x,t) = f * g_t(x),$$

where

$$g_t(x) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}$$

(the proof of this lies in **Proposition 8.24** [4]). So, in some sense, we've found that the solution must be of this form.

1.3. **When do things work.** In the last section, we disregarded essentially all rigor to try to find a solution. In this section, we want to go back and try to figure out where the conditions are right for us to get our desired solution (i.e., when can we drop the formal when we take the formal Fourier transform).

Let's start with the most rigorous space, Schwartz space. Take $f \in \mathcal{S}$. We can examine the solution to the PDE $u(x,t)$ in two different contexts: in the first (the standard way) we have $u(x,t) : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ where $(x,t) \mapsto u(x,t)$. In the second interpretation, we can take $u : [0,\infty) \to \mathcal{S}$ – that is, u maps t to some kind of Schwartz function.[2]

One issue in the second interpretation is that we don't know what continuity and derivatives look like. We try to extend their definitions now.

Remark. From here on out, we switch our definition of Schwartz norm. We now define the norms via

$$\|f\|_{(\alpha,\beta)} = \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta f(x)|.$$

We note that the topology generated by these norms is equivalent to the topology generated by the norms given by

$$\|f\|_{(N,\alpha)} = \sup_{x\in\mathbb{R}^n} (1+|x|)^N |\partial^\alpha f(x)|$$

by **Proposition 8.3** [4], so there is no issue in taking either interpretation.

Definition. Note that convergence in $\mathcal{S}$ is with respect to the Schwartz norms.

(1) Let $(a,b) \subset \mathbb{R}$ be some open interval. Then the function $u : (a,b) \to \mathcal{S}$ is said to be *continuous at t* if $u(t+h) \to u(t)$ in $\mathcal{S}$ as $h \to 0$.

(2) We say that u is *differentiable at t* if

$$\frac{u(t+h)-u(t)}{h} \to v(t) \text{ in } \mathcal{S}$$

as $t \to 0$, where $v(t) \in \mathcal{S}$ is some function. Note that we call v the *strong derivative of u*, and we denote it by $u_t(t) = v(t)$.

[2]This is apparently important in PDE.

One thing to remark here is that we can extend the first definition to half open intervals and closed intervals in the obvious way: for $u : [a, b) \to \mathcal{S}$ to be continuous, we would require that $u(a + h) \to u(a)$ in $\mathcal{S}$ as $h \to 0$, and likewise for $u : (a, b] \to \mathcal{S}$ to be continuous, we would require that $u(b - h) \to u(b)$ in $\mathcal{S}$ as $h \to 0$. Likewise, we can extend the definition of continuity at a point to continuity on an interval in the obvious way: a function is said to be continuous if it is continuous at all points in the interval.

We denote by $C([a, b], \mathcal{S})$ the set of $\mathcal{S}$ continuous functions.

Proposition (Proposition 5.3 [3]**).** Suppose that $u \in C([a, b], \mathcal{S})$, and $u(t) = u(\cdot, t)$. Then $u \in C^1((a, b), \mathcal{S})$ iff:

(1) The pointwise derivatives $\partial_t u(x, t)$ exist for all $x \in \mathbb{R}^n$, $t \in (a, b)$.
(2) $\partial_t u(\cdot, t) \in \mathcal{S}$ for all $t \in (a, b)$.
(3) The map $t \mapsto \partial_t u(\cdot, t)$ belongs to $C((a, b), \mathcal{S})$.

Proof. ($\Longrightarrow$): First, assume that $u \in C^1((a, b), \mathcal{S})$. We wish to show that conditions (1)-(3) hold. Since $u \in C^1((a, b), \mathcal{S})$, we have that u is strongly differentiable at each point $t \in (a, b)$; that is, there is some function $v \in \mathcal{S}$ so that for all (α, β), as $h \to 0$,

$$\left\| \frac{u(t+h) - u(t)}{h} - v(t) \right\|_{(\alpha,\beta)} \to 0.$$

Taking α and β to be 0, we get that this translates to

$$\left\| \frac{u(t+h) - u(t)}{h} - v(t) \right\|_{u} \to 0$$

as $h \to 0$. In other words, there is a pointwise derivative for u at each point $t \in (a, b)$, so that $\partial_t u(x, t)$ exists for all $x \in \mathbb{R}^n$, $t \in (a, b)$. This satisfies condition (1). Notice as well that this implies that $\partial_t u(\cdot, t) = v(t) \in \mathcal{S}$ for all $t \in (a, b)$. This satisfies condition (2). Finally, for condition (3) we simply note that $u \in C^1((a, b), \mathcal{S})$ implies that the derivative is in $C((a, b), \mathcal{S})$; that is, $v(t) = \partial_t u(\cdot, t)$ is such that $v \in C((a, b), \mathcal{S})$. Thus, all three conditions are satisfied.
($\Longleftarrow$): Assume conditions (1)-(3) are satisfied. Our goal is to show that $\partial_t u(\cdot, t) \in C((a, b), \mathcal{S})$. That is, our goal is to show that there exists a $v \in C((a, b), \mathcal{S})$ so that for all (α, β),

$$\left\| \frac{u(t+h) - u(t)}{h} - v(t) \right\|_{(\alpha,\beta)} \to 0$$

as $h \to 0$. Notice that for each $x \in \mathbb{R}^n$, we have that

$$\begin{aligned} \frac{u(x, t+h) - u(x, t)}{h} - \partial_t u(x, t) &= \frac{1}{h} \int_t^{t+h} \partial_s u(x, s) ds - \partial_t u(x, t) \\ &= \frac{1}{h} \int_t^{t+h} [\partial_s u(x, s) - \partial_t u(x, t)] ds, \end{aligned}$$

using the fundamental theorem of Calculus. Note that this is valid to do by the properties. Furthermore, we have that it is well-defined by the properties.

We note as well that for each (α, β), we have

$$x^\alpha \partial^\beta \left[\frac{u(x, t+h) - u(x, t)}{h} - \partial_t u(x, t) \right] = x^\alpha \partial^\beta \frac{1}{h} \int_t^{t+h} [\partial_s u(x, s) - \partial_t u(x, t)] ds.$$

We wish to now bring the derivative on the inside of the integral. Noting that the inside of the integral is bounded by an L^1 function, we can induct **Theorem 2.27 (b)** [4] to get that

$$x^\alpha \partial^\beta \left[\frac{u(x, t+h) - u(x, t)}{h} - \partial_t u(x, t) \right] = \frac{1}{h} \int_t^{t+h} x^\alpha \partial^\beta [\partial_s u(x, s) - \partial_t u(x, t)] ds.$$

Alternatively, as the lecture notes say, we could just directly use DCT. Now, applying the alternative definition of the Schwartz seminorm, we have

$$\begin{aligned}\left\|\frac{u(\cdot,t+h)-u(\cdot,t)}{h}-\partial_t u(\cdot,t)\right\|_{(\alpha,\beta)} &= \sup_{x\in\mathbb{R}^n}\left|\frac{1}{h}\int_t^{t+h} x^\alpha\partial^\beta[\partial_s u(x,s)-\partial_t u(x,t)]ds.\right| \\ &\le \frac{1}{|h|}\int_t^{t+h}\sup_{x\in\mathbb{R}^n}|x^\alpha\partial^\beta[\partial_s u(x,s)-\partial_t u(x,t)]ds| \\ &= \frac{1}{|h|}\int_t^{t+h}\|\partial_s u(\cdot,s)-\partial_t u(\cdot,t)\|_{(\alpha,\beta)}ds \\ &\le \max_{t\le s\le t+h}\|\partial_s u(\cdot,s)-\partial_t u(\cdot,t)\|_{(\alpha,\beta)}.\end{aligned}$$

We note now that $\partial_t u \in C((a,b),\mathcal{S})$ by property (3), and so taking the limit as $h\to 0$ of both sides gives us

$$\lim_{h\to 0}\left\|\frac{u(\cdot,t+h)-u(\cdot,t)}{h}-\partial_t u(\cdot,t)\right\|_{(\alpha,\beta)}=0.$$

In other words, we have that $\partial_t u(\cdot,t)$ is the strong derivative of u as well, so that $u\in C^1((a,b),\mathcal{S})$. □

With this proposition, we can make some of what we claimed in 1.2 rigorous. That is, if we assume that the initial condition f is in the Schwartz space, then we get our desired solution.

Theorem (Schwartz Solution (**Theorem 5.4** [3]))**.** Suppose $f\in\mathcal{S}$. Then there is a (unique)[3] solution $u\in C([0,\infty),\mathcal{S})\cap C^1((0,\infty),\mathcal{S})$ of the heat equation. Furthermore, we will get that $u\in C^\infty((0,\infty),\mathcal{S})$, and the Fourier transform of the solution is given by

$$\widehat{u}(m,t)=\widehat{f}(m)e^{-t4\pi^2|m|^2}.$$

For $t>0$, this will give us that the solution is

$$u(x,t)=\int f(y)\Gamma(x-y,t)dy,$$

where

$$\Gamma(x,t)=\frac{1}{(4\pi t)^{n/2}}e^{-|x|^2/(4t)}.$$[4]

Proof. The proof is given by a series of claims. We recall that $\mathcal{F}$ and $\mathcal{F}^{-1}$ are continuous linear maps on $\mathcal{S}$ (**Corollary 8.23** [4]).

Claim (Property 1)**.** We have that $\partial_t u$ exists iff $\partial_t\widehat{u}$ exists. Furthermore, we have

$$\mathcal{F}(\partial_t u)=\partial_t(\mathcal{F}u).$$

Proof. ($\Longrightarrow$) Assume $\partial_t u$ exists. The existence of ∂_t implies that for all (α,β), we have

$$\left\|\frac{u(\cdot,t+h)-u(\cdot,t)}{h}-\partial_t u(\cdot,t)\right\|_{(\alpha,\beta)}\to 0$$

as $h\to 0$. The idea, then, is to apply the Fourier transform here. Doing so, we have

$$\left\|\frac{\widehat{u}(\cdot,t+h)-\widehat{u}(\cdot,t)}{h}-\widehat{\partial_t u(\cdot,t)}\right\|_{(\alpha,\beta)}.$$

[3]We will not comment on why there is uniqueness in the proof.

[4]We call this *Green's function.* This is *not* the Gamma function.

We wish to show, then, that this goes to 0 as $h \to 0$ for all (α, β). Fix an $\epsilon > 0$. Since the Fourier transform is continuous on Schwartz space, we have that for all $\epsilon > 0$, there exists a $\delta > 0$ so that

$$\|u\|_{(\alpha,\beta)} < \delta \implies \|\widehat{u}\|_{(\alpha,\beta)} < \epsilon.$$

Since the original sequence goes to 0, we have that there exists h sufficiently small so that

$$\left\| \frac{\widehat{u}(\cdot, t+h) - \widehat{u}(\cdot, t)}{h} - \partial_t \widehat{u(\cdot, t)} \right\|_{(\alpha,\beta)} < \delta,$$

where δ is such that

$$\left\| \frac{\widehat{u}(\cdot, t+h) - \widehat{u}(\cdot, t)}{h} - \partial_t \widehat{u(\cdot, t)} \right\|_{(\alpha,\beta)} < \epsilon.$$

Since we can do this for all $\epsilon > 0$, we get that we can let $\epsilon \to 0$ to get

$$\left\| \frac{\widehat{u}(\cdot, t+h) - \widehat{u}(\cdot, t)}{h} - \partial_t \widehat{u(\cdot, t)} \right\|_{(\alpha,\beta)} \to 0$$

as $h \to 0$. We did this for arbitrary (α, β), and so the Schwartz limit is going to 0; i.e., $\partial_t \widehat{u(\cdot, t)} = \widehat{\partial_t u(\cdot, t)}$.
$(\impliedby)$: The argument is exactly the same, except now we use the continuity of $\mathcal{F}^{-1}$. □

Claim (Property 2)**.** We have that $u \in C([0, \infty), \mathcal{S})$ if and only if $\widehat{u} \in C([0, \infty), \mathcal{S})$.

Proof. $(\implies)$: Assume that $u \in C([0, \infty), \mathcal{S})$. Then we have that for all (α, β),

$$\|u(\cdot, t+h) - u(\cdot, t)\|_{(\alpha,\beta)} \to 0$$

as $h \to 0$. We again do the same argument; applying the Fourier transform, we see that we get

$$\|\widehat{u}(\cdot, t+h) - \widehat{u}(\cdot, t)\|_{(\alpha,\beta)},$$

and we fix some $\epsilon > 0$. Continuity of the Fourier transform says that there is some $\delta > 0$ such that

$$\|u(\cdot, t+h) - u(\cdot, t)\|_{(\alpha,\beta)} < \delta \implies \|\widehat{u}(\cdot, t+h) - \widehat{u}(\cdot, t)\|_{(\alpha,\beta)} < \epsilon,$$

and since

$$\|u(\cdot, t+h) - u(\cdot, t)\|_{(\alpha,\beta)} \to 0,$$

we can take h sufficiently small so that

$$\|\widehat{u}(\cdot, t+h) - \widehat{u}(\cdot, t)\|_{(\alpha,\beta)} < \epsilon.$$

Since $\epsilon > 0$ arbitrary, we get the desired result.
$(\impliedby)$: Again, the same argument, except now use the continuity of $\mathcal{F}^{-1}$. □

Claim (Property 3)**.** We have that $u \in C^k((0, \infty), \mathcal{S})$ iff $\widehat{u} \in C^k((0, \infty), \mathcal{S})$ for all k.

Proof. $(\implies)$: We proceed by induction. The case $k = 0$ was shown in the last claim (essentially), and so we assume that it holds for $k - 1$. We wish to show that $u \in C^k((0, \infty), \mathcal{S})$ implies that $\widehat{u} \in C^k((0, \infty), \mathcal{S})$. The induction hypothesis tells us that we have $\widehat{u} \in C^{(k-1)}((0, \infty), \mathcal{S})$, and so we need to show that $\partial_t^{(k-1)} \widehat{u}(\cdot, t) \in C^1((0, \infty), \mathcal{S})$. But notice that, since the Fourier transform and derivative commute, we get that the pointwise derivative $\partial_t^{(k)} \widehat{u}(\cdot, t) = \widehat{\partial_t^{(k)} u(\cdot, t)}$ exists, it is a Schwartz function for all t, and it is in $C((0, \infty), \mathcal{S})$ by continuity of the Schwartz function (see the prior two arguments). Hence, by **Proposition 1.3**, we get that $\widehat{u} \in C^{(k)}((0, \infty), \mathcal{S})$.
$(\impliedby)$: Again, the same kind of argument. □

We have that these three claims gives us that $u(x,t)$ is a solution, $u(x,t) \in C([0,\infty),\mathcal{S}) \cap C^1((0,\infty),\mathcal{S})$ iff $\widehat{u}$ is a solution to the ODE given by taking the Fourier transform of the PDE; i.e., iff $\widehat{u}$ is a solution to

$$\partial_t \widehat{u}(m,t) = -4\pi^2 |m|^2 \widehat{u}(m,t),$$

with initial condition $\widehat{u}(m,0) = \widehat{f}(m)$, $\widehat{u} \in C([0,\infty),\mathcal{S}) \cap C^1((0,\infty),\mathcal{S})$.

First, if $\widehat{u} \in C([0,\infty),\mathcal{S}) \cap C^1((0,\infty),\mathcal{S})$, then we see that **Proposition 1.3** tells us that $\widehat{u}(m,t)$ is pointwise-differentiable with respect to t in $t > 0$ and continuous in $t \geq 0$ for each fixed m. Solving the ODE[5] gives

$$\widehat{u}(m,t) = \widehat{f}(m) e^{-4\pi^2 t|m|^2}.$$

The ODE uniqueness and existence theorem tell us that this is a unique solution.

Now, we wish to show that $\widehat{u}(m,t) = \widehat{f}(m)e^{-4\pi t|m|^2}$ (the spatial Fourier transform of the solution) is strongly differentiable with derivative

$$\partial_t \widehat{u}(m,t) = -4\pi^2 |m|^2 \widehat{f}(m) e^{-4\pi^2 t|m|^2}.$$

Let (α,β) be any multi-indices. Then

$$\begin{aligned} m^\alpha \partial^\beta [\widehat{u}(m,t+h) - \widehat{u}(m,t)] &= m^\alpha \partial^\beta \widehat{u}(m,t+h) - m^\alpha \partial^\beta \widehat{u}(m,t) \\ &= m^\alpha \partial^\beta \left[\widehat{f}(m) e^{-4\pi(t+h)|m|^2}\right] - m^\alpha \partial^\beta \left[\widehat{f}(m) e^{-4\pi t|m|^2}\right] \\ = \widehat{a}(m,t)\left[e^{-4\pi h|m|^2} - 1\right] e^{-4\pi t|m|^2} &+ h \sum_{j=0}^{|\beta|-1} h^j \widehat{b}_j(m,t) e^{-4\pi(t+h)|m|^2}, \end{aligned}$$

where $\widehat{a}(\cdot,t)$ $\widehat{b}_j(\cdot,t) \in \mathcal{S}$. Taking the supremum over m, we see that the above equation implies that

$$\|\widehat{u}(t+h) - \widehat{u}(t)\|_{(\alpha,\beta)} \to 0$$

as $h \to 0$. So $\widehat{u} \in C([0,\infty),\mathcal{S})$. A similarly tedious argument gives us that $\partial_t \widehat{u}(\cdot,t)$ is continuous, so applying **Proposition 1.3** we get that $\widehat{u}$ is strongly continuous.

Finally, **Proposition 8.24** [4] tells us that

$$\mathcal{F}^{-1}(e^{-4\pi^2 t|m|^2}) = \mathcal{F}^{-1}(e^{-\pi(4\pi t)|m|^2}) = (4\pi t)^{-n/2}\, e^{-|x|^2/(4t)}.$$

So we have that

$$u(x,t) = \int f(y)\Gamma(x-y,t)dy,$$

where

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/(4t)}.$$

□

One thing to remark is that in the lecture notes we switched definitions of Fourier transform to be consistent with [3]. The issue is that this is inconsistent with Folland's definition. As a result, I have tried to change the formulas to be consistent with Folland, and consequently we see that our Green function is not defined in the same way.

Next, we note that the restriction of $f \in \mathcal{S}$ is pretty strong. We can actually weaken this to $f \in L^p$ (i.e. we explore the question of what happens when f is not smooth).

[5] Look at prior section for details.

Theorem (Smoothing (**Theorem 5.5** [3])). Suppose that $p \in [1, \infty)$, $f \in L^p$. Define $u : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$ by

$$u(x,t) = \int f(y)\Gamma(x-y,t)dy.$$

Then $u \in C_0^\infty(\mathbb{R}^n \times (0,\infty))$ and $\partial_t u = \nabla^2 u$ in $t > 0$. Furthermore, we have that $u(\cdot, t) \to f$ in L^p as $t \to 0^+$.

Proof. First, notice that Γ is such that

$$\partial_t \Gamma = \nabla^2 \Gamma,$$

which can be seen with a (tedious) calculation.

Remark. If you're comfortable with this, you can use the Dirac delta function as an initial condition. Doing so will show you this equality without having to do any tedious calculations, since this implies that

$$u(x,t) = \int \delta(y)\Gamma(x-y,t)dy = \Gamma(x,t)$$

is a solution to the heat equation. However we offer no justification for why you may do this.

Notice as well that $\Gamma(\cdot, t) \in L^q$ for all $q \in [1, \infty]$, together with all of its derivatives. To see this, fixing $t > 0$, we have

$$\|\Gamma(\cdot,t)\|_q^q = \int \left| \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/(4t)} \right|^q dx = \frac{1}{(4\pi t)^{qn/2}} \int e^{-q|x|^2/(4t)} dx.$$

Using **Proposition 2.53** [4], we get that this is finite. Now, take arbitrary derivatives of $\Gamma(x,t)$ with respect to x; i.e., examine $\partial^\alpha \Gamma(x,t)$. We have

$$\begin{aligned}
\|\partial^\alpha \Gamma(\cdot,t)\|_q^q &= \int \left| \partial^\alpha \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/(4t)} \right|^q dx \\
&= \frac{1}{(4\pi t)^{n/2}} \int \left| \partial^\alpha e^{-|x|^2/(4t)} \right| dx \\
= \frac{1}{(4\pi t)^{n/2}} &\int \left| \partial^\alpha e^{-x_1^2/(4t)} \cdots e^{-x_n^2/(4t)} \right| dx.
\end{aligned}$$

Notice here that we can iterate the integral (via Tonelli), and this problem reduces to examining whether

$$\int \left| P(x^2) e^{-x^2/(4t)} \right| dx,$$

where P is a polynomial. The linearity of the integral and the triangle inequality reduces this problem to examining whether

$$\int |x^{2n} e^{-x^2/(4t)}| dx$$

has finite integral for $n \geq 0$. However, this is just a simple application of polar coordinates again, via using **Proposition 2.53** [4]. Iterating the integral, we get that each of these are finite, and so we have that the function is in L^q, as desired.

We want to now take derivatives with respect to x and t under the integral. We examine that

$$\frac{d}{dx} u(x,t) = \frac{d}{dx} \int f(y)\Gamma(x-y,t)dy.$$

That is, we have the pointwise derivative is given by

$$\lim_{h \to 0} \frac{1}{h} \int f(y)[\Gamma(x+h-y,t) - \Gamma(x-y,t)]dy.$$

We see that the inside is bounded by

$$|f(y)||\Gamma(x+h-y,t)-\Gamma(x-y,t)|,$$

which we see is in L^1 since integrating this with respect to y and noting that we can examine $h \leq 1$ (since we're taking $h \to 0$) to get

$$\int |f(y)||\Gamma(x+h-y,t)-\Gamma(x-y,t)|dy \leq \|f\|_p \max_{x \leq s \leq x+1} \|\Gamma(s-\cdot,t)-\Gamma(x-\cdot,t)\|_q < \infty,$$

so we can DCT to bring in the limit and get the derivative of Γ. Inducting gives us that it holds for all number of derivatives of x. A similar argument gives us that it holds for derivatives with t as well, and so we see we can move all derivatives under the integral. Moreover, these derivatives all go to 0 as $|x| \to \infty$, since it holds for Γ. Thus, u is a smooth, decaying solution of the heat equation for $t > 0$. We see that Γ is a "mollifier," (similar to approximate identity), so we see that $\Gamma_t * f \to f$ as $t \to 0$ in L^p. □

As a final remark, we note that we can use Riesz-Thorin to get a nice estimate on the q norm for $2 \leq q \leq \infty$.

Theorem (Estimates (**Theorem 5.8** [3])). Let $u : [0,\infty) \to \mathcal{S}$ be the solution of the heat equation constructed prior. Then

$$\|u(t)\|_2 \leq \|f\|_2, \quad \|u(t)\|_\infty \leq \frac{1}{(4\pi t)^{n/2}}\|f\|_1,$$

and furthermore we have for all $2 \leq q \leq \infty$,

$$\|u(t)\|_q \leq \frac{1}{(4\pi t)^{n(1/2-1/p)}}\|f\|_p,$$

where $(p,q) = 1$.

Proof. To see the first inequality, we have that Plancherel gives us

$$\|u(t)\|_2 = \|\widehat{u(t)}\|_2 = \left\|\widehat{f}e^{-4\pi^2|k|^2t}\right\|_2 \leq \|\widehat{f}\|_2 = \|f\|_2.$$

To see the second, notice that for all x we have

$$|u(x,t)| = \left|\int \Gamma(x-y,t)f(y)dy\right| \leq \|\Gamma(x-\cdot,t)f\|_1 \leq \|\Gamma(\cdot,t)\|_\infty\|f\|_1$$

by Hölders inequality. Notice that

$$|\Gamma(x,t)| = \left|\frac{1}{(4\pi t)^{n/2}}e^{-|x|^2/(4t)}\right| = \frac{1}{(4\pi t)^{n/2}},$$

so

$$\|\Gamma(\cdot,t)\|_\infty = \frac{1}{(4\pi t)^{n/2}},$$

giving us

$$\|u(t)\|_\infty \leq \|u(t)\|_u \leq \frac{1}{(4\pi t)^{n/2}}\|f\|_1,$$

as desired.

To see the final part, we invoke Riesz-Thorin. Recall that it says the following:

Theorem (**Theorem 6.27** [4])**.** Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are measure spaces and $p_0, p_1, q_0, q_1 \in [1, \infty]$. For $0 < t < 1$, define p_t and q_t by

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}, \qquad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}.$$

If T is a linear map from $L^{p_0}(\mu) + L^{p_1}(\mu)$ into $L^{q_0}(\nu) + L^{q_1}(\nu)$ such that $\|Tf\|_{q_0} \leq M_0\|f\|_{p_0}$ for $f \in L^{p_0}(\mu)$, $\|Tf\|_{q_1} \leq M_1\|f\|_{p_1}$ for $f \in L^{p_1}(\mu)$, then $\|Tf\|_{q_t} \leq M_0^{1-t}M_1^t\|f\|_{p_t}$ for $f \in L^{p_t}(\mu)$, $0 < t < 1$.

Thus, setting $q_0 = 2$, $q_1 = \infty$, $M_0 = 1$, $M_1 = 1/(4\pi t)^{n/2}$, then we have that

$$\|u(t)\|_q \leq \frac{1}{(4\pi t)^{r(n/2)}}\|f\|_p,$$

where $(p, q) = 1$ and we replace the t in the theorem with an r (to remove confusion). Notice as well we have

$$\frac{1}{q} = \frac{1-r}{q_0} + \frac{r}{q_1} = \frac{1-r}{2} \implies \frac{q-2}{q} = r.$$

Substituting this in gives

$$\|u(t)\|_q \leq \frac{1}{(4\pi t)^{n(1/2-1/q)}}\|f\|_p,$$

as desired. □

2. Laplace Transform

2.1. **What is the Laplace transform.** We define the Laplace transform for $s \in \mathbb{C}$ by

$$\mathcal{L}[f](s) = \int_0^\infty e^{-st}f(t)dt.$$

Let $F(t) \in \mathrm{BV}$; i.e., we have

$$\lim_{x\to\infty} T_F(x) = \sup\left\{\sum_1^n |F(x_i) - F(x_{i-1})| : n \in \mathbb{N}, -\infty < x_0 < \cdots < x_n = x\right\} < \infty.$$

One thing to note is that we also have $\mathrm{Re}(F), \mathrm{Im}(F) \in \mathrm{BV}$.[6] Let $s \in \mathbb{C}$. Then for any $R > 0$, we can define

$$\int_0^R e^{-st}dF(t),$$

where this is a Stieltjes integral; i.e., this is the integral defined by the limit

$$\int_a^b f(t)d\alpha(t) = \lim_{\delta\to 0}\sum_{i=0}^{n-1} f(\xi_i)[\alpha(x_{i+1}) - \alpha(x_i)],$$

with $\alpha(x_i) \leq \xi_i \leq \alpha(x_{i+1})$, and the limit exists independently of the manner of subdivision and choices of ξ.

We define then the improper integral via

$$\int_0^\infty e^{-st}dF(t) = \lim_{R\to\infty}\int_0^\infty e^{-st}dF(t),$$

where we note that we have fixed $s \in \mathbb{C}$. If the limit exists, we say that the integral converges for the value of s. This leads us to our first convergence theorem.

[6]This follows from the triangle inequality.

Theorem (**Theorem 2.1** [5], pg. 36)**.** If

$$\sup_{u\geq 0}\left|\int_0^u e^{-s_0 t}dF(t)\right| = M < \infty,$$

then

$$\int_0^\infty e^{-st}dF(t)$$

converges for $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > \mathrm{Re}(s_0)$, and

$$\int_0^\infty e^{-st}dF(t) = (s - s_0)\int_0^\infty e^{-(s-s_0)t}G(t)dt,$$

where

$$G(t) = \int_0^u e^{-s_0 t}dF(t).$$

Proof. We do integration by parts, noting that

$$\int_0^R e^{-st}dF(t) = \int_0^R e^{-(s-s_0)t}dG(t)$$

$$= e^{-(s-s_0)R}G(r) + (s - s_0)\int_0^R e^{-(s-s_0)t}G(t)dt,$$

and we see that applying the limit as $R \to \infty$ gives

$$\lim_{R\to\infty} e^{-(s-s_0)R}G(R) = 0$$

for $\mathrm{Re}(s) > \mathrm{Re}(s_0)$, since

$$G(R) = \int_0^R e^{-s_0 t}dF(t),$$

and by assumption we have

$$\lim_{R\to\infty} G(R) = \int_0^\infty e^{-s_0 t}dF(t) \leq M,$$

so that the limit of the product goes to 0. Thus, we have

$$\int_0^\infty e^{-st}dF(t) = (s - s_0)\int_0^\infty e^{-(s-s_0)t}G(t)dt.$$

□

Notice that implicitly, we used the following lemma:

Lemma (**Theorem 1.6 (b)** [5], pg. 12)**.** If f and φ are continuous and α is BV, and if

$$d\beta(t) = \varphi(t)d\alpha(t),$$

in other words,

$$\beta(t) = \int_c^t \varphi(\tau)d\alpha(\tau),$$

then

$$\int_a^b f(t)d\beta(t) = \int_a^b f(t)\varphi(t)d\alpha(t).$$

The theorem gives us the following corollary:

Corollary. If

$$\int_0^\infty e^{-st} dF(t)$$

converges for $\mathrm{Re}(s) = \sigma_0$, then it converges for all s such that $\mathrm{Re}(s) > \sigma_0$.

2.2. **Popular Laplace transforms.** One immediate issue with the Laplace transform (over the Fourier transform) is that the inverse is not great to work with. We can, in some ad hoc fashion, construct a table of potential functions to work with in order to try to find the inverse.

Example. We list off a few popular Laplace transforms and give "justifications."

(1) The linearity of the Laplace transform is clear by the linearity of integral. That is, if $a, b \in \mathbb{R}$, f, g are functions where the Laplace transform is defined, then we have

$$\mathcal{L}[af + bf](s) = \int_0^\infty (af + bg)(t) e^{-st} dt$$
$$= a \int_0^\infty f(t) e^{-st} dt + b \int_0^\infty g(t) e^{-st} dt = a\mathcal{L}[f](s) + b\mathcal{L}[g](s).$$

(2) Let

$$u(t) = \begin{cases} 1 \text{ if } t > 0 \\ 0 \text{ if } t < 0. \end{cases}$$

Then we see that

$$\mathcal{L}[u](s) = \int_0^\infty e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_{t=0}^{\infty} = \frac{1}{s},$$

so long as $\mathrm{Re}(s) > 0$. Hence, we have $\mathcal{L}[1](s) = \frac{1}{s}$

(3) We have

$$\mathcal{L}[e^{at}](s) = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{t(a-s)} dt$$
$$= \frac{1}{a-s} e^{(a-s)t} \Big|_{t=0}^{\infty} = \frac{1}{s-a}$$

so long as $\mathrm{Re}(a - s) < 0$, or $\mathrm{Re}(a) < \mathrm{Re}(s)$.

(4) We have

$$\mathcal{L}[f'](s) = \int_0^\infty e^{-st} f'(t) dt,$$

and doing integration by parts, we let $dv = f'(t)dt$, $v = f(t)$, $u = e^{-st}$, $du = -se^{-st}dt$. Then we have

$$\mathcal{L}[f'](s) = f(t)e^{-st} \Big|_{t=0}^{\infty} + s\mathcal{L}[f](s) = -f(0+) + s\mathcal{L}[f](s),$$

where we assume that f is nice enough for things to work (which we necessarily needed for the transform to exist in the first place; i.e. we need that f doesn't blow up faster than an exponential).

(5) We can induct this sort of strategy. Thus, we have the following claim:

Claim. Assuming nice enough conditions on f, we have that

$$\mathcal{L}[f^{(n)}](s) = s^n \mathcal{L}[f](s) - s^{n-1} f(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0).$$

Proof. We established this for the case $n = 1$, so assume it holds up to $k-1$. Then we have

$$\mathcal{L}[f^{(k)}](s) = \int e^{-st} f^{(k)}(t)dt.$$

Let $dv = f^{(k)}(t)dt$, $v = f^{(k-1)}(t)$, $u = e^{-st}$, $du = -se^{-st}dt$. Then we have

$$\begin{aligned}\mathcal{L}[f^{(k)}](s) &= e^{-st} f^{(k-1)}(t)\Big|_{t=0}^{\infty} + s\mathcal{L}[f^{(k-1)}](s) \\ &= -f^{(k-1)}(0) + s\left(s^{k-1}\mathcal{L}[f](s) - s^{k-2}f(0) - \cdots - f^{(k-2)}(0)\right) \\ &= -f^{(k-1)}(0) + s^k\mathcal{L}[f](s) - s^{k-1}f(0) - \cdots - sf^{(k-2)}(0),\end{aligned}$$

as desired. □

(6) For f and g which admit Laplace transforms (i.e. assume that they are nice enough so things converge absolutely), we have

$$\mathcal{L}[f * g](s) = \mathcal{L}[f](s)\mathcal{L}[g](s).$$

To see this, we have

$$\begin{aligned}\mathcal{L}[f * g](s) &= \int_0^\infty e^{-st}(f * g)(t)dt \\ &= \int_0^\infty e^{-st}\left[\int_0^\infty f(t-x)g(x)dx\right]dt \\ &= \int_0^\infty \int_0^\infty e^{-st} f(t-x)g(x)dxdt \\ &= \int_0^\infty \int_0^\infty e^{-st} f(t-x)g(x)dtdx \\ &= \int_0^\infty e^{-sx}g(x)\int_0^\infty e^{-s(t-x)}f(t-x)dtdx \\ &= \mathcal{L}[f](s)\mathcal{L}[g](s),\end{aligned}$$

where the justification of switching the order of integration comes from first applying Tonelli and then applying Fubini (where we assumed that things converge absolutely).

(7) Notice that we have

$$\mathcal{L}[u(t-a)f(t)](s) = e^{-as}\mathcal{L}[\tau_{-a}f](s).$$

To prove this, we first establish

$$\mathcal{L}[u(t-a)f(t-a)](s) = e^{-as}\mathcal{L}[f](s).$$

To see this, we have

$$\begin{aligned}\mathcal{L}[u(t-a)f(t-a)](s) &= \int_0^\infty u(t-a)f(t-a)e^{-st}dt \\ &= \int_a^\infty f(t-a)e^{-st)}dt \\ &= \int_0^\infty f(t)e^{-st}e^{-sa} = \mathcal{L}[f](s)e^{-sa}.\end{aligned}$$

To deduce the second result, let $g(t) = f(t+a)$, then $f(t) = g(t-a)$ and

$$\mathcal{L}[u(t-a)f(t-a)](s) = \mathcal{L}[u(t-a)g(t)](s) = e^{-sa}\mathcal{L}[g(t)](s) = e^{-sa}\mathcal{L}[f(t+a)](s),$$

as desired.

(8) By (1), we see that

$$\mathcal{L}[1](s) = \mathcal{L}[u](s) = \frac{1}{s}.$$

We claim then that

$$\mathcal{L}[t^n](s) = \frac{n}{s}\mathcal{L}[t^{n-1}](s).$$

To see this, we have

$$\mathcal{L}[t^n](s) = \int_0^\infty t^n e^{-st} dt.$$

Let $u = t^n$, $du = nt^{n-1}dt$, $dv = e^{-st}dt$, $v = -(1/s)e^{-st}$. Then we have that this is equal to

$$\mathcal{L}[t^n](s) = -\frac{1}{s}t^n e^{-st}\bigg|_{t=0}^{\infty} + \frac{n}{s}\int_0^\infty e^{-st}t^{n-1}dt = \frac{n}{s}\mathcal{L}[t^{n-1}](s).$$

So inducting gives

$$\mathcal{L}[t^n](s) = \frac{n!}{s^{n+1}}.$$

(9) Recall by DeMoivre we have

$$e^{it} = \cos(t) + i\sin(t).$$

By (3) and (1), we get

$$\mathcal{L}[e^{it}](s) = \mathcal{L}[\cos(t)](s) + i\mathcal{L}[\sin(t)](s) = \frac{1}{s-i}.$$

We can write

$$\frac{1}{s-i} = \frac{s+i}{s^2+1} = \frac{s}{s^2+1} + \frac{i}{s^2-1},$$

and so

$$\mathcal{L}[\sin(t)](s) = \frac{1}{s^2+1},$$
$$\mathcal{L}[\cos(t)](s) = \frac{s}{s^2+1}.$$

In general, we can deduce that

$$e^{iat} = \cos(at) + i\sin(at),$$
$$\mathcal{L}[e^{iat}](s) = \frac{1}{s-ia} = \frac{s+ia}{s^2+a^2} = \frac{s}{s^2+a^2} + \frac{ia}{s^2+a^2},$$

so

$$\mathcal{L}[\sin(at)](s) = \frac{a}{s^2+a^2},$$
$$\mathcal{L}[\cos(at)](s) = \frac{s}{s^2+a^2}.$$

(10) Consider the Dirac delta function (which is not really a function). That is, the linear functional δ so that

$$\int \delta(t)f(t)dt = f(0).$$

Then we see that

$$\mathcal{L}[\delta](s) = \int_0^\infty \delta(t)e^{-st}dt = e^{-s(0)} = 1.$$

Furthermore, consider translations, we see that

$$\mathcal{L}[\tau_y\delta](s) = \int_0^\infty \delta(t-y)e^{-st}dt.$$

Note we can write this as

$$\mathcal{L}[\tau_y \delta](s) = \int_0^\infty \delta(t-y) e^{-s(t-y)} e^{-sy} dt = e^{-sy},$$

assuming $y > 0$.

2.3. **Applications of Laplace transform to IVP.** We use the Laplace transform to solve some IVP.

Example. Consider the ODE

$$y'' - 10y' + 9y = 5t, \quad y(0) = -1, \; y'(0) = 2.$$

We apply the Laplace transform to get

$$\mathcal{L}[y''](s) - 10\mathcal{L}[y'](s) + 9\mathcal{L}[y](s) = 5\mathcal{L}[t](s).$$

Now use (5) and (8) from the prior example to get

$$\mathcal{L}[t](s) = \frac{1}{s^2},$$
$$\mathcal{L}[y''](s) = s^2\mathcal{L}[y](s) - sy(0) - y'(0),$$
$$\mathcal{L}[y'](s) = s\mathcal{L}[y](s) - y(0).$$

Applying the initial conditions, this gives

$$\mathcal{L}[y''](s) = s^2\mathcal{L}[y](s) + s - 2,$$
$$\mathcal{L}[y'](s) = s\mathcal{L}[y](s) + 1,$$

so substituting this in, we get

$$s^2\mathcal{L}[y](s) + s - 2 - 10s\mathcal{L}[y](s) - 10 + 9\mathcal{L}[y](s) = \frac{5}{s^2}.$$

Simplifying, we have

$$(s^2 - 10s + 9)\mathcal{L}[y](s) + s - 12 = \frac{5}{s^2}.$$

We now solve for $\mathcal{L}[y](s)$ to get

$$\mathcal{L}[y](S) = \frac{5 - s^3 + 12s^2}{s^2(s-1)(s-9)}.$$

We now do partial fractions. Recall that we will have

$$\frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s-9} = \frac{5 - s^3 + 12s^2}{s^2(s-1)(s-9)}.$$

Clearing denominators, this gives

$$A(s(s-1)(s-9)) + B(s-1)(s-9) + C(s^2(s-9)) + D(s^2(s-1)) = 5 - s^3 + 12s^2.$$

Let $s = 0$ we get

$$9B = 5 \implies B = \frac{5}{9},$$

letting $s = 1$, we get

$$-8C = 16 \implies C = -2,$$

letting $s = 9$, we get

$$648D = 248 \implies D = \frac{31}{81},$$

and finally, substituting all these in, we have

$$A(s(s-1)(s-9)) + \frac{5}{9}(s-1)(s-9) - 2(s^2(s-9)) + \frac{31}{81}(s^2(s-1)) = 5 - s^3 + 12s^2,$$

we let $s = 2$ and solve to get

$$A = \frac{50}{81}.$$

Thus, we have

$$\mathcal{L}[y](s) = \frac{50}{s81} + \frac{5}{9s^2} + \frac{31}{81(s-9)} - \frac{2}{s-1}.$$

We now need to take the inverse Laplace transform. We can use the linearity of this to simplify things greatly. Recall that $\mathcal{L}[1](s) = \frac{1}{s}$, so we have

$$\mathcal{L}^{-1}[(50/81)s] = \frac{50}{81},$$

we have

$$\mathcal{L}[t] = \frac{1}{s^2},$$

so

$$\mathcal{L}^{-1}[(5/9)s^2] = \frac{5}{9}t,$$

we have

$$\mathcal{L}[e^{9t}](s) = \frac{1}{s-9},$$

so

$$\mathcal{L}^{-1}[(31/(81(s-9))] = \frac{31}{81}e^{9t},$$

and finally

$$\mathcal{L}[e^t](s) = \frac{1}{s-1},$$

so we have

$$\mathcal{L}^{-1}[-2/(s-1)] = -2e^t.$$

Hence, we have

$$y = \frac{50}{81} + \frac{5}{9}t + \frac{31}{81}e^{9t} - 2e^t.$$

Going back, the Laplace transform can even be useful to solving the heat equation assuming everything is nice enough.

Example. We will take the Laplace transform with respect to the time variable. Hence, letting $\widehat{u}(m,t) = y(m,t)$ (for notational simplicity), we have the initial value problem

$$\partial_t y(m,t) = -4\pi^2|m|^2 y(m,t).$$

Taking the Laplace transform, we get

$$\mathcal{L}[\partial_t y(m,t)](s) = \int_0^\infty \partial_t y(m,t)e^{-st}dt.$$

Let $dv = \partial_t y(m,t)dt$, $v = y(m,t)$, $u = e^{-st}$, $du = -se^{-st}dt$, then

$$\mathcal{L}[\partial_t y(m,t)](s) = e^{-st}y(m,t)\bigg|_{t=0}^{\infty} + s\int_0^\infty y(m,t)e^{-st}dt = -y(m,0) + s\mathcal{L}[y(m,t)](s).$$

Recall that the initial condition says $y(m,0) = \widehat{f}(m)$, and we assume $\widehat{f}(m)$ is Schwartz (since f is Schwartz), so this is really

$$-\widehat{f}(m) + s\mathcal{L}[y(m,t)](s) = \mathcal{L}[\partial_t y(m,t)](s).$$

Thus, our ODE gives us

$$s\mathcal{L}[y(m,t)](s) - \widehat{f}(m) = -4\pi^2|m|^2\mathcal{L}[y(m,t)](s),$$

or

$$\mathcal{L}[y(m,t)](s) = \frac{\widehat{f}(m)}{s + 4\pi^2|m|^2}.$$

Taking the inverse Laplace transform, we recall (3) to get

$$y(m,t) = \widehat{u}(m,t) = \widehat{f}(m)e^{-4\pi^2|m|^2}.$$

Notice this aligns with what we had before. Taking the inverse Fourier transform, we get

$$u(m,t) = f * g_t(x),$$

where

$$g_t(x) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/(4t)}.$$

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